A STUDY OF CCD LATTICES IN A FUNCTOR CATEGORY

by

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Abstract

In their monograph "An Extension of the Galois Theory of Grothendieck", Joyal and Tierney characterized sup lattices and locales in a functor category $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. In the early 1990's, Fawcett and Wood introduced a new lattice structure which is a special case of locales: constructively completely distributive (CCD) lattices. This work brings together these two concepts by attempting to characterize CCD lattices in a functor category.

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Chapter 1

Introduction

The purpose of this thesis is to study constructively completely distributive (CCD) lattices in a functor category $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. In Joyal and Tierney's monograph "An Extension of the Galois Theory of Grothendieck" [4], the authors characterize sup lattices and locales in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. As we shall see, CCD lattices are a special case of locales. Thus, a characterization of CCD lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ is a natural extension to Joyal and Tierney's work.

In Chapter 2, we review the main points of CCD theory, discussed in the following papers: Fawcett and Wood [1] and Rosebrugh and Wood [6], [7], [8]. This chapter emphasizes the use of down objects ($\mathcal{D}$) instead of power objects ($\mathcal{P}$) as the codomain for the Yoneda map. This is an integral aspect of CCD theory, and is the key difference between CCD lattices and completely distributive (CD) lattices. The main result of the chapter is Theorem 2.3.4, where a characterization of sup lattices, locales and CCD lattices in terms of adjoints to the down mapping is given.

In Chapter 3, we review the features of the topos $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ neccesary to investigate poset structures. While most of this material is well-known, it is useful both as a review and and to help standardize notation before the structures are used extensively in Chapter 4. The features investigated in this chapter are $\mathcal{P}$, $\mathcal{D}$, posets, and the down mapping.

In Chapter 4, we determine the nature of sup lattices, locales, and CCD lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. As mentioned above, the first two structures have previously been characterized in Joyal and Tierney [4], with another proof given in Johnstone [3]. In this thesis, we approach the characterization of these structures in a different manner than that given in those two works, where several external results are used. The proofs contained here all follow a similar internal pattern. To determine the nature of L, a poset in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ with some additional structure, we exploit the fact that $\mathcal{D}L$ is always a sup lattice, locale, and CCD lattice. There are a couple of advantages to this method: it does not require any external results, and the idea of looking at $\mathcal{D}L$ is a method that could be used in other toposes.

Unfortunately, the characterization of CCD lattices given in Chapter 4 assumes that the category $\mathbf{C}$ has wide pullbacks. Obviously, this is a very large assumption, as it essentially forces $\mathbf{C}$ to be a poset. In Chapter 5, we attempt to find a way around this assumption. To do this, we first attempt to remove the assumption that $\mathbf{C}$ has pullbacks from the characterization of sup lattices. This is accomplished by working in a more 2-categorical fashion. The chapter ends with a characterization of sup lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ for any small category $\mathbf{C}$, then a conjecture for a characterization of CCD lattices.

In the final chapter, we note several directions that could be taken for further research in this area.

Chapter 2

The General Theory of CCD Lattices

For ease of explication, in this chapter we will work as though the base topos is **set**, though everything we say works in any topos **E**.

2.1 The Power Object and the Down Object

The importance of the power object of an object L, written $\mathcal{P}L$, is well-known. In fact, the existence of a power object for each object in a category, together with the existence of finite limits, is equivalent to the category being a topos (Johnstone [2, p.92]). However, for objects L which also carry a poset structure $\leq$, the down object, $\mathcal{D}(L, \leq)$, is even more important than $\mathcal{P}L$. Normally, we will assume the existence of $\leq$ and simply speak of $\mathcal{D}L$.

In **set**, $\mathcal{D}L$ is defined as:

$$\mathcal{D}L = \{B \subseteq L : x \leq y \text{ and } y \in B \Rightarrow x \in B\}$$

In other words, $\mathcal{D}L$ is the set of down-closed subsets of L. This definition is constructive, and so we can interpret it in any topos **E**.

To understand the down object in a bit more detail, we will first investigate the univeral property of a power object, and then show how the down object has a similar universal property. We will frame both of these universal properties in the language of relations.

Definition Let A and B be objects in **E**. A *relation* R between A and B, written $B \overset{R}{\not\longrightarrow} A$, is a subobject $R \hookrightarrow A \times B$ [1]. Denote the category whose morphisms are relations of **E** by **rel**(**E**), with composition given by image of pullback.

[1]Note that as in Rosebrugh and Wood [8], we write the relation "backwards" to agree with the convention for profunctors.

In **set**, the composition sr of the relations $A \overset{r}{\nrightarrow} B$ and $B \overset{s}{\nrightarrow} C$ is given by, for $c \in C$ and $a \in A$,

$$\frac{c(sr)a}{\exists b \in B : c(s)b(r)a}$$

Example 2.1.1 *Since $\mathcal{P}L = \Omega^L$, we have the following evaluation map:*

$$L \times \mathcal{P}L \xrightarrow{ev} \Omega$$

Define $\in$ as the subobject corresponding to ev:

$$\in \rightarrowtail L \times \mathcal{P}L$$

that is, a relation

$$\mathcal{P}L \overset{\in}{\nrightarrow} L$$

which, in **set***, is the standard membership relation.*

Definition Given a morphism $C \xrightarrow{f} D$ in $\mathbf{E}$, we can form the *graph* of f, denoted by $C \overset{f_*}{\nrightarrow} D$, as the relation given by the monomorphism $C \xrightarrow{(f, 1_C)} D \times C$. That is, $\forall c \in C$, $\forall d \in D$,

$$d(f_*)c \text{ if } d = f(c)$$

We can also form the *op-graph* of f, $D \overset{f^*}{\nrightarrow} C$, given by

$$c(f^*)d \text{ if } f(c) = d$$

The universal property of the power object can now be stated in a simple form:

Proposition 2.1.2 *(Universal Property of the Power Object)*
If L is an object of $\mathbf{E}$*, then the power object $\mathcal{P}L$, together with the membership relation $\mathcal{P}L \overset{\in}{\nrightarrow} L$, has the following universal property: for each relation $B \overset{R}{\nrightarrow} L$, there exists a unique morphism $B \xrightarrow{r} \mathcal{P}L$ such that the following commutes in* **rel***:*

$$\begin{array}{ccc} B & \overset{r_*}{\nrightarrow} & \mathcal{P}L \\ & {\scriptstyle R}\searrow\!\!\!\!\!/ & \Big\downarrow\!\!\!\!/ \; {\scriptstyle \in} \\ & & L \end{array}$$

r is known as the name *of the relation R.*

Proof Johnstone [2, p.68-69].

Example 2.1.3 *For* **set**, *given a relation* $B \overset{R}{\nrightarrow} L$, *r is the map* $b \mapsto \{a \in L : aRb\}$.

For objects which carry a poset structure, a special type of relation is more useful. Let **ord**(**E**) denote the category of internal posets of **E** and functors between them. Just as morphisms in **E** give graph relations $()_*$ and $()^*$, morphisms in **ord**(**E**) give "ordered graph" relations:

Definition Given $C \xrightarrow{f} D$ in **ord**, define $C \overset{f_*}{\nrightarrow} D$, the *ordered graph of* f, by

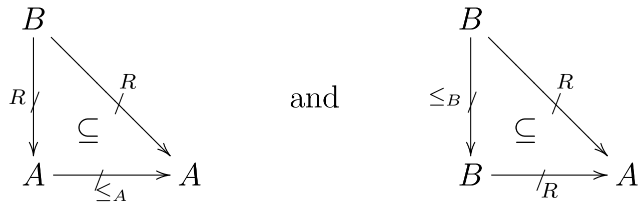
$$d(f_*)c \text{ if } d \leq f(c)$$

Similarly, define the *op-ordered graph of* f, $D \overset{f^*}{\nrightarrow} C$ by

$$c(f^*)d \text{ if } f(c) \leq d$$

These relations, however, have an additional property: they are order-ideal relations.

Definition For $A, B \in$ **ord**(**E**), an *order ideal relation* R is a relation $B \overset{R}{\nrightarrow} A$ such that



In **set**, these amount to the conditions

$$a \leq b(R)c \Rightarrow a(R)c$$
$$a(R)b \leq c \Rightarrow a(R)c$$

Then, as mentioned above, we have:

Proposition 2.1.4 *If* $C \xrightarrow{f} D$ *is a morphism in* **ord**(**E**), *then the ordered graph* f_* *and the op-ordered graph* f^* *are order-ideal relations.*

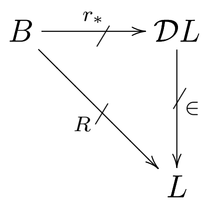
Proof For the first equation, suppose $a \leq d(f_*)c$. Then $a \leq d \leq f(c)$, so $a \leq f(c)$, thus $a(f_*)c$.

For the other equation, suppose $d(f_*)c \leq a$. Then $d \leq f(c) \leq f(a)$ since f is order-preserving. Hence $d \leq f(a)$, so $d(f_*)a$.

A similar proof shows f^* is also an order ideal. ■

We are now in a position to state the universal property of the down object $\mathcal{D}L$.

Proposition 2.1.5 *(Universal Property of the Down Object)*
If L is a poset of **E**, *then the down object $\mathcal{D}L$, together with the restriction of the membership relation $\mathcal{D}L \overset{\in}{\nrightarrow} L$, has the following universal property: for each order ideal relation $B \overset{R}{\nrightarrow} L$, there exists a unique order-preserving morphism $B \xrightarrow{r} \mathcal{D}L$ such that:*



we will call r the ordered name *of the order ideal relation R.*

Proof We can prove this by using the universal property for P. Since R is a relation, there exists a morphism $B \xrightarrow{r} \mathcal{P}L$ which factors through $\mathcal{P}L \overset{\in}{\nrightarrow} L$. If we can show that the fact that R is an order-ideal relation implies that r maps into $\mathcal{D}L$ and is order-preserving, then the commutivity of the above triangle and uniqueness will follow by the commutivity and uniqueness in Proposition 2.1.2.

To show r maps into $\mathcal{D}L$, we need, for $b \in B$, $r(b)$ to be down-closed. Indeed:

$$\begin{aligned} & a_1 \leq a_2 \in r(b) \\ \Rightarrow\ & a_1 \leq a_2(R)b \\ \Rightarrow\ & a_1(R)b \text{ (the first order-ideal property)} \\ \Rightarrow\ & a_1 \in r(b) \end{aligned}$$

To show r is order-preserving, we need $b_1 \leq b_2$ to imply $r(b_1) \subseteq r(b_2)$. Suppose we have:

$$b_1 \leq b_2 \text{ and } a \in r(b_1)$$

$$\Rightarrow \quad a(R)b_1 \leq b_2$$
$$\Rightarrow \quad a(R)b_2 \text{ (the second order-ideal property)}$$
$$\Rightarrow \quad a \in r(b_2)$$

Thus the two order ideal properties imply that $\mathcal{D}$ has the required universal property.■

As we shall see, this universal property is one method of obtaining the important $\downarrow$ mapping.

2.2 The Down Mapping ($\downarrow$)

There are two equally important ways of constructing the $\downarrow$ mapping. The first uses the universal property in Proposition 2.1.5, the second is the Yoneda embedding.

Definition For a poset L in **E**, define the $L \xrightarrow{\downarrow} \mathcal{D}L$ morphism to be the ordered name of the relation $L \overset{\leq}{\not\longrightarrow} L$.

The reason for the $\downarrow$ notation is the following:

Example 2.2.1 *In* **set***, the* $\downarrow$ *mapping is given by:* $a \mapsto \{b : b \leq a\}$*, ie. the* $\downarrow$ *mapping takes* a *to the set of all elements less than or equal to* a*.*

As we shall see, $\downarrow$ is the basis for the theory of both sup lattices and CCD lattices. Another reason for its importance is that it is an instance of the Yoneda Embedding.

Let L be a poset. As an internal poset, the hom-sets of L lie in Ω. In other words, L is an Ω-enriched internal category. The Yoneda Embedding for the category L is:

$$\begin{array}{rcl} L & \longrightarrow & \Omega^{L^{\mathrm{op}}} \\ x & \longmapsto & L(-,x) \end{array}$$

where the exponent is taken in $\mathbf{ord}(\mathbf{E})$. However, $\Omega^{L^{\mathrm{op}}}$ has a more specific form:

Lemma 2.2.2 $\Omega^{L^{op}} \cong \mathcal{D}L$*.*

Proof Every $L^{\mathrm{op}} \xrightarrow{F} \Omega$ defines a subobject of L^{op}, say $\Psi(F) \hookrightarrow L^{\mathrm{op}}$. We need to check $\Psi(F)$ is down-closed. Suppose $y \leq x$, and $x \in \Psi(F)$. Since $y \leq x$, $F(x) \leq F(y)$. Let 1 denote the top element in Ω. Then $x \in \Psi(F) \Rightarrow F(x) = 1$, so $F(y) = 1$, hence $y \in \Psi(F)$. Moreover,

$$\frac{\frac{\frac{F \subseteq G}{F(x) \leq G(x)\ \forall x \in L}}{x \in F^{-1}(1) \Rightarrow x \in G^{-1}(1)}}{\Psi(F) \subseteq \Psi(G)}$$

■

Proposition 2.2.3 *The maps $L \xrightarrow{\downarrow} \mathcal{D}L$ and the Yoneda Embedding $L \longrightarrow \Omega^{L^{op}}$ are identified by the isomorphism of Lemma 2.2.2.*

Proof The Yoneda Embedding is $x \mapsto L(-, x)$. Now:

$$\frac{\frac{\frac{y \in L(-,x)}{y \in L(-,x)^{-1}(1)}}{L(y,x) = 1}}{y \leq x}$$

So indeed $L(-, x) = \downarrow(x)$. ■

The Yoneda Lemma gives a useful fact about $\mathcal{D}L$:

Proposition 2.2.4 *For L a poset, $x \in L, F \in \mathcal{D}L$, the Yoneda lemma for L states that $(x \in F) \Leftrightarrow (\downarrow x \subseteq F)$. In particular, $(x \leq y) \Leftrightarrow (\downarrow x \subseteq \downarrow y)$.*

Proof The Yoneda lemma states that

$$\frac{F(x)}{L(-,x) \longrightarrow F}$$

Which, rewritten by the equivalences mentioned above, is nothing more than

$$\frac{x \in F}{\downarrow x \subseteq F}$$

■

2.3 Sup Lattices, Locales, and CCD Lattices

We will first define each of the poset properties, then show how they relate to $\downarrow$.

Definition A *sup lattice* is a poset L such that for any down-closed subset $M \in \mathcal{D}L$, there exists an element of L, $\bigvee M$, such that $x \in M \Rightarrow x \leq \bigvee M$, and $\bigvee M$ is universal with this property, ie. $\forall y \in L$ such that $x \in M \Rightarrow x \leq y$, we have $\bigvee M \leq y$.

Then every sup lattice L has an **ord** arrow $\mathcal{D}L \xrightarrow{\bigvee} L$. The following result is well-known:

Proposition 2.3.1 *If L is a sup lattice, then L^{op} is also a sup lattice, in other words, any sup lattice is also an inf lattice.*

Proof See Section 2.6. ■

A locale is a sup lattice with an extra distributivity condition.

Definition A sup lattice L is a *locale* if we have $\forall x \in L, M \in \mathcal{D}L$,

$$x \wedge \bigvee M = \bigvee \{x \wedge m : m \in M\}$$

Finally, we get to the more recent property, constructively completely distributive (CCD) lattices, first defined in Fawcett and Wood [1]. The idea of a CCD lattice is simply an infinite extension of the idea of locales.

Definition A sup lattice L is a *CCD lattice* if we have $\forall \mathcal{F} \subseteq \mathcal{D}L$,

$$\bigwedge \left\{ \bigvee S : S \in \mathcal{F} \right\} = \bigvee \left\{ \bigwedge \{T(S) : S \in \mathcal{F}\} : T \in \Pi\mathcal{F} \right\}$$

where $\Pi\mathcal{F}$ represents the set of all choice functions for $\mathcal{F}$. This definition can be simplified by the following lemma:

Lemma 2.3.2 *Suppose L is a sup lattice, and $\mathcal{F} \subseteq \mathcal{D}L$. Then*

$$\left\{ \bigwedge \{T(S) : S \in \mathcal{F}\} : T \in \Pi\mathcal{F} \right\} = \bigcap \mathcal{F}$$

Proof " $\supseteq$ " Suppose $x \in \bigcap \mathcal{F}$. Define $X \in \prod \mathcal{F}$ by $X(S) = x$.
Then $\bigwedge\{X(S) : S \in \mathcal{F}\} = \bigvee\{x : S \in \mathcal{F}\} = x$, so x is in the left side of the equation.

" $\subseteq$ " Suppose now that x is in the left side. Then there is some $T \in \prod \mathcal{F}$ such that $x = \bigwedge\{T(S) : S \in \mathcal{F}\}$. So for any $S \in \mathcal{F}$, $x \leq T(S) \in S$. Then S down-closed implies $x \in S$, so $x \in \bigcap \mathcal{F}$. ■

Corollary 2.3.3 *If L is a poset, L is a CCD lattice if and only if $\forall \mathcal{F} \subseteq \mathcal{D}L$,*

$$\bigwedge\left\{\bigvee S : S \in \mathcal{F}\right\} = \bigvee\left(\bigcap \mathcal{F}\right)$$

In Section 2.5 we will we investigate the differences between this new idea and the classical notion of completely distributive (CD) lattices. In particular, we will see why we use $\mathcal{D}L$ instead of $\mathcal{P}L$. For now, we have the following key theorem, which relates each of the above structures to left adjoints of $L \xrightarrow{\downarrow} \mathcal{D}L$:

Theorem 2.3.4 *If L is a poset, then:*

1. *L is a sup lattice $\Leftrightarrow \downarrow$ has a left adjoint $(\bigvee)$.*

2. *L is a locale $\Leftrightarrow \downarrow$ has a left adjoint $(\bigvee)$ which preserves binary meets.*

3. *L is a CCD lattice $\Leftrightarrow \downarrow$ has a left adjoint $(\bigvee)$ which itself has a left adjoint.*

This characterization allows one to see how natural a progression the idea of a CCD lattice is from the ideas of sup lattice and locale.

Proof (of part 1 of Theorem 2.3.4)

($\Rightarrow$) Suppose L has a sup operation, $\bigvee : \mathcal{D}L \to L$. Claim that $\bigvee \dashv \downarrow$. Indeed:

$$\frac{\dfrac{\bigvee M \leq x}{\forall m \in M, m \leq x}}{M \subseteq \downarrow x}$$

One should also note that while the adjunction $\bigvee \dashv \downarrow$ gives $\bigvee \downarrow x \leq x$, we actually have $\bigvee \downarrow x = x$, since it is always true that $x \leq \bigvee \downarrow x$.

($\Leftarrow$) Suppose $\downarrow$ has a left adjoint, say $S \dashv \downarrow$. Then we have:

$$\begin{aligned} & x \in M \\ \Rightarrow\ & \downarrow x \subseteq M \text{ (lemma 2.2.4)} \\ \Rightarrow\ & S \downarrow x \leq SM \\ \Rightarrow\ & \downarrow x \subseteq \downarrow (SM) \text{ (adjunction)} \\ \Rightarrow\ & x \leq SM \text{ (lemma 2.2.4)} \end{aligned}$$

So $\forall x \in M$, $x \leq SM$.

Now suppose that $\forall x \in M$, $x \leq y$. Then $M \subseteq \downarrow y$, so by the adjunction, $SM \leq y$. Thus $\bigvee M = SM$ exists.

Before proving part 2, we need a couple of lemmas:

Lemma 2.3.5 *For $N \in \mathcal{D}L$, $x \in L$, $(\downarrow x) \cap N = \{x \wedge n : n \in N\}$*

Proof ($\subseteq$) Suppose $n \in (\downarrow x) \cap N$. Then $n \leq x$, so $n = x \wedge n$.

($\supseteq$) Suppose $y = x \wedge n$ for $n \in N$. Then $y \leq x$, so $y \in (\downarrow x)$, and $y \leq n$, so N is down-closed implies $y \in N$. ■

Lemma 2.3.6 *For $M, N \in \mathcal{D}L$, $M \cap N = \{m \wedge n : m \in M, n \in N\}$.*

Proof ($\subseteq$) Suppose $x \in M \cap N$. Then since $x = x \wedge x$, x is in the RS of the equation.

($\supseteq$) Suppose $x = m \wedge n$. Then $x \leq m$, so $x \in M$, and $x \leq n$, so $x \in N$. ■.

We can now prove Part 2.

Proof (Part 2 of Theorem 2.3.4)

($\Rightarrow$) Suppose L is a locale. Then we have:

$$\begin{aligned} & (\bigvee M) \wedge (\bigvee N) \\ =\ & \bigvee \left\{ \left(\bigvee M \right) \wedge n : n \in N \right\} \text{ (locale property)} \\ =\ & \bigvee \left\{ \bigvee \{m \wedge n : m \in M\} : n \in N \right\} \text{ (locale property)} \\ =\ & \bigvee \{m \wedge n : m \in M, n \in N\} \\ =\ & \bigvee (M \cap N) \text{ (lemma 2.3.6)} \end{aligned}$$

So indeed $\bigvee$ preserves binary meets.

($\Leftarrow$) Suppose $\bigvee$ preserves $\wedge$. Then:

$$\begin{aligned} & m \wedge \left(\bigvee N\right) \\ = & \left(\bigvee \downarrow m\right) \wedge \left(\bigvee N\right) \\ = & \bigvee(\downarrow m \ \cap N) \text{ (by assumption)} \\ = & \bigvee\{m \wedge n : n \in N\} \text{ (by lemma 2.3.5)} \end{aligned}$$

which says that L is a locale.

And finally, the last part of the proof of Theorem 2.3.4:

Proof (Part 3 of Theorem 2.3.4)

($\Rightarrow$) Suppose L is a CCD lattice. Define:

$$\begin{aligned} \Downarrow : L & \longrightarrow \mathcal{D}L \\ x & \longmapsto \{y : y \ll x\} \end{aligned}$$

where $y \ll x \Leftrightarrow \forall M \in \mathcal{D}L, x \leq \bigvee M \Rightarrow y \in M$.

It is easy to check that $\Downarrow$ is actually in **ord**. We now want to show $\Downarrow \dashv \bigvee$. The co-unit of the adjunction, $\Downarrow \bigvee M \subseteq M$, follows directly from the definition of $\Downarrow$. Indeed, $x \in \Downarrow \bigvee M$ means that $x \ll \bigvee M$, and hence $\bigvee M \leq \bigvee M$ implies $x \in M$.

For the unit, $x \leq \bigvee \Downarrow x$, define a family

$$\mathcal{F}_x = \{S \in \mathcal{D}L : x \leq \bigvee S\}$$

Then $\Downarrow x$ is the intersection of the family $\mathcal{F}_x$:

$$\dfrac{\dfrac{\dfrac{\dfrac{y \in \Downarrow x}{y \ll x}}{\forall M \in \mathcal{D}L, x \leq \bigvee M \Rightarrow y \in M}}{\forall S \in \mathcal{F}_x, y \in S}}{y \in \bigcap \mathcal{F}_x}$$

Hence

$$\begin{aligned} & \bigvee \Downarrow x \\ = \; & \bigvee \bigcap \mathcal{F}_x \\ = \; & \bigwedge \{ \bigvee S : S \in \mathcal{F}_x \} \text{ (since L is CCD)} \\ \geq \; & x \text{ (by definition of } \mathcal{F}_x) \end{aligned}$$

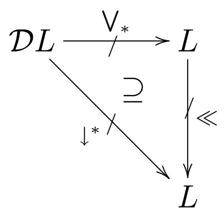
So the adjunction holds.

($\Leftarrow$) Suppose we have a map $L \xrightarrow{\Downarrow} \mathcal{D}L$ in **ord** which is left adjoint to $\bigvee$. Then $\bigvee$ preserves all limits, in other words, $\bigvee$ preserves all infima. Then by the alternate CCD equation (Corollary 2.3.3), L is a CCD lattice. ■

2.4 The $\ll$ Relation and the $\Downarrow$ Map

In the proof of Part 3 of Theorem 2.3.4, we introduced a new relation: the $\ll$ relation, and defined $\Downarrow$ to be its ordered name. Here we collect a few facts about these new concepts.

First, note that the $\ll$ relation exists in any sup lattice, since it is defined in terms of $\bigvee$. Diagramatically, it can be expressed as the right Kan extension of $\downarrow^*$ along $\bigvee_*$:



Indeed, the relation $\ll$ given by the diagram above says

$$\begin{array}{c} y \ll x \\ \hline \forall M \in \mathcal{D}L, (x(\bigvee_*)M \Rightarrow y(\downarrow^*)M) \\ \hline \forall M \in \mathcal{D}L, (x \leq \bigvee M \Rightarrow \downarrow y \subseteq M) \\ \hline \forall M \in \mathcal{D}L, (x \leq \bigvee M \Rightarrow y \in M) \end{array}$$

which is how we defined $\ll$ in Theorem 2.3.4. Moreover, since $\ll$ is the Kan extension of these order-ideal relations, it is itself an order ideal. If $y \ll x$, we say that y is *totally below* x.

The following result justifies the use of the $\Downarrow$ notation, and the name "totally below":

Proposition 2.4.1 *For L a sup lattice, and $x \in L$, we have* $\Downarrow x \subseteq \downarrow x$*, that is, $y \ll x$ implies $y \leq x$.*

Proof Suppose $y \ll x$.
Taking $M = \downarrow x$, since $x \leq \bigvee \downarrow x$, we must have $y \in \downarrow x$. ■

A counter-example shows that the converse does not hold. Consider the lattice $L = \{0, a, b, c, 1\}$, with $0 \leq a, b, c \leq 1$. We have $a \leq 1$, but $a \not\ll 1$, since $\bigvee\{b, c\} = 1$, but $a \notin \{b, c\}$.

2.5 CCD vs CD

The definition of complete distributivity, CD, is as follows: $\forall \mathcal{F} \subseteq \mathcal{P}L$,

$$\bigwedge\left\{\bigvee S : S \in \mathcal{F}\right\} = \bigvee\left\{\bigwedge\{T(S) : S \in \mathcal{F}\} : T \in \Pi\mathcal{F}\right\}$$

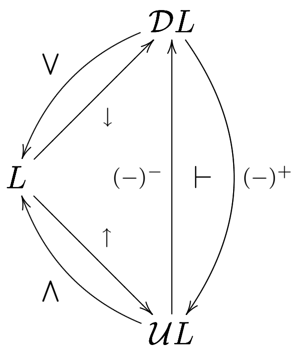
Thus the definitions of CD and CCD are identical, except for the fact that CD requires $\forall \mathcal{F} \subseteq \mathcal{P}L$, while CCD requires $\forall \mathcal{F} \subseteq \mathcal{D}L$. Why use $\mathcal{D}$ and CCD instead of $\mathcal{P}$ and CD? The most compelling argument is the fact that $\forall L \in$ **ord**, both $\mathcal{D}L$ and $\mathcal{P}L$ are CCD, while $\mathcal{P}L$ or $\mathcal{D}L$ being CD requires the axiom of choice (Fawcett and Wood [1]). Obviously, having the "set of subsets" object being distributive is a very valuable property. Since most toposes (in particular, $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$), do not have the axiom of choice, one should use CCD instead of CD in an arbitrary topos.

Moreover, using $\mathcal{D}$ and CCD instead of $\mathcal{P}$ and CD is more aesthetically pleasing from a category theory point of view, because of Theorem 2.3.4, which characterizes these structures in terms of adjoints. While Part 1 of Theorem 2.3.4 is still true if we replace $\mathcal{D}$ with $\mathcal{P}$, Parts 2 and 3 are not. Thus, by using only power sets, one must define locales and CD lattices in terms of indexed equations. Using down sets, one can define locales and CCD lattices in terms of properties of a left adjoint to the Yoneda Embedding $L \xrightarrow{\downarrow} \mathcal{D}L$.

2.6 The Up Functor $\mathcal{U}$

While we have been focusing on the down functor $\mathcal{D}$, we also have the up functor $\mathcal{U}$. $\mathcal{U}L$ is the set of up-closed subsets of L, ordered by *reverse* inclusion. This means that if L is a sup lattice, then the up map, $L \xrightarrow{\uparrow} \mathcal{U}L$, is *left* adjoint to $\bigwedge$. So, for a sup lattice L, we have the adjoint strings $\bigvee \dashv \downarrow$ and $\uparrow \dashv \bigwedge$. We will use these concepts later when determining the nature of sup lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$.

Additionally, these adjoint strings demonstrate how sup lattices are inf lattices, and vice versa (Fawcett and Wood [1]):



where M^+ is the set of upper bounds of M, and N^- is the set of lower bounds of N. Then the existence of $\bigvee$ gives $\bigwedge$ by defining $\bigwedge = \bigvee(-)^-$, and similarly one can define $\bigvee = \bigwedge(-)^+$.

Chapter 3

The Topos $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$

In this section we will detail the key elements of the topos $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ which we need to understand various poset structures in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. As we have seen in Chapter 2, the down object is a key element of CCD theory. To understand the down object in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, we must first investigate the power object.

3.1 Subobjects

In **set**, $\mathcal{P}$ of an object is simply the set of its subsets. In $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, we shall see that the notion of subobjects is also a key element of the power object, so it is useful to determine what subojects are in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. We will use the following notation:

Notation For $F, G \in \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, say that F is a *subfunctor* of G, and write $F \subseteq G$, if F is a subobject of G.

We then have the following characterization of a subfunctor:

Proposition 3.1.1

$$F \textit{ is a subfunctor of } G \iff \begin{array}{l} \textit{(i) } \forall C \in \mathbf{C}, F(C) \subseteq G(C) \\ \textit{(ii) } \forall f \in \mathbf{C}, F(f) = G(f) \textit{ restricted to } F(C). \end{array}$$

Proof ($\Rightarrow$) If F is a subfunctor of G, then we have some monomorphism $F \xrightarrow{i} G$. Then each function $F(C) \xrightarrow{i_C} G(C)$ is also a mono, so $F(C) \subseteq G(C)$.

Let $D \xrightarrow{f} C \in \mathbf{C}$. Then naturality of i implies

$$\begin{array}{ccc} F(C) & \overset{i_C}{\rightarrowtail} & G(D) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ G(C) & \underset{i_D}{\rightarrowtail} & G(D) \end{array}$$

commutes, which says that $F(f)$ equals $G(f)$ restricted to $F(C)$.

($\Leftarrow$) Since $F(C) \subseteq G(C)$, for each C, we have a monomorphism $F(C) \xrightarrow{i_C} G(C)$. Define $F \xrightarrow{i} G$ by making its components the i_C maps. Then i is natural by (ii), and a monomorphism since each i_C is a monomorphism. ■

This proposition gives us an effective and useful method of constructing subobjects in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Given a functor G, to define a subfunctor F, we will simply define each $F(C)$ as some subset of $G(C)$, then check that for each f, $G(f)$ restricted to $F(C)$ maps into $F(D)$. By the above proposition, if that holds, then F will indeed be a subfunctor of G.

3.2 The Power Object

We would like to determine for an object $L \in \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ the nature of its power object $\mathcal{P}L$. To do this, we will use the Yoneda Lemma to find what $\mathcal{P}L = \Omega^L$ must be, define a suitable $\in$ relation, then verify that $\mathcal{P}L$ with this membership relation has the universal property of a power object seen in Proposition 2.1.2.

Consider the following string of equivalences:

$$\begin{array}{c} \mathcal{P}L(C) \\ \hline \Omega^L(C) \\ \hline \mathbf{C}(-,C) \longrightarrow \Omega^L \\ \hline \mathbf{C}(-,C) \times L \longrightarrow \Omega \\ \hline \hookrightarrow \mathbf{C}(-,C) \times L \end{array}$$

Thus $\mathcal{P}L(C) = \{F \in \mathbf{set}^{\mathbf{C}^{\mathrm{op}}} : F \subseteq \mathbf{C}(-,C) \times L\}$. That is, $\mathcal{P}L(C)$ is the set of subfunctors of $\mathbf{C}(-,C) \times L$.

Now, given $D \xrightarrow{f} C$, we also need to define $\mathcal{P}L(f)$. That is, given $F \in \mathcal{P}L(C)$, we need to define $\mathcal{P}L(f)(F)$, a subfunctor of $\mathbf{C}(-,D) \times L$. Given an object E of $\mathbf{C}$, define

$$\mathcal{P}L(f)(F)(E) = \{(g,y) \in [E,D] \times L(E) : (fg,y) \in F(E)\}$$

As noted in Section 3.1, we need to check that this actually defines a subfunctor of $\mathbf{C}(-,D) \times L$. That is, given $E_2 \xrightarrow{k} E_1$, and $(g,y) \in \mathcal{P}L(f)(F)(E_1)$, we need

$(gk, L(k)(e)) \in \mathcal{P}L(f)(F)(E_2)$. Indeed,

$$\begin{aligned}(fg, y) \in F(E_1) &\Rightarrow F(k)(fg, y) \in F(E_2)\\ &\Rightarrow (fgk, L(k)(y)) \in F(E_2)\\ &\Rightarrow (gk, L(k)(y)) \in \mathcal{P}L(F)(f)(E_2)\end{aligned}$$

So we have defined how $\mathcal{P}L$ acts on objects and arrows of $\mathbf{C}$.

Proposition 3.2.1 *For each $L \in \mathbf{C}$, $\mathcal{P}L$ as defined above is an object of $\mathbf{set}^{\mathbf{C}^{op}}$.*

Proof We have checked that our definition of $\mathcal{P}L$ is well-defined on objects and arrows. That $\mathcal{P}L$ maps identities to identities is obvious. So, we only need to check that if we have $E \xrightarrow{g} D$ and $D \xrightarrow{f} C$ in $\mathbf{C}$, then $\mathcal{P}L(g)\mathcal{P}L(f) = \mathcal{P}L(fg)$. Indeed, for each $H \in \mathcal{P}L(E)$, $M \in \mathbf{C}$:

$$\cfrac{\cfrac{\cfrac{(h, z) \in \mathcal{P}L(fg)(H)(M)}{(fgh, z) \in H(M)}}{(gh, z) \in \mathcal{P}L(f)(H)(M)}}{(h, z) \in \mathcal{P}L(g)\mathcal{P}L(f)(H)(M)}$$

So indeed $\mathcal{P}L$ is a functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{set}$. ■

Now, to check that the universal property holds, we need to define a membership relation $\mathcal{P}L \overset{\in}{\longrightarrow\!\!\!\!\!/} L$. So we need for each $C \in \mathbf{C}$, a relation $\mathcal{P}L(C) \overset{\in_C}{\longrightarrow\!\!\!\!\!/} L(C)$. Given $F \in PL(C)$, $x \in L(C)$, we'll try the following:

$$x(\in_C)F \iff (1_C, x) \in F(C)$$

Again, for $\in$ to be a subobject, we need to check that (ii) of Proposition 3.1.1 holds. That is, given an $f \in \mathbf{C}$, we need to have

$$x(\in_C)F \Rightarrow L(f)(x)[\in_D]\mathcal{P}L(f)(F)$$

However, this follows since

$$\begin{aligned}x(\in_C)F &\Rightarrow (1_C, x) \in F(C)\\ &\Rightarrow (f, L(f)(x)) \in F(D)\\ &\Rightarrow (1_D, L(f)(x)) \in \mathcal{P}L(f)(F)\\ &\Rightarrow L(f)(x)[\in_D]\mathcal{P}L(f)(F)\end{aligned}$$

Thus for each object L of $\mathbf{C}$, we have a power object $\mathcal{P}L$ and a membership relation $\mathcal{P}L \stackrel{\in}{\nrightarrow} L$. All we need to do now is check that the universal property is satisfied. The proof of this universal property also gives us the name (see Proposition 2.1.2) of a relation in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$.

Proposition 3.2.2 *The power-object functor $\mathcal{P}$ for $\mathbf{set}^{\mathbf{C}^{op}}$ is given on objects by:*

$$\mathcal{P}L(C) = \{F \in \mathbf{set}^{\mathbf{C}^{op}} : F \subseteq \mathbf{C}(-, C) \times L\}$$

For an arrow $D \xrightarrow{f} C$ in $\mathbf{C}$, and $F \in \mathcal{P}L(C)$, $\mathcal{P}L(f)$ is defined by:

$$\mathcal{P}L(f)(F)(E) = \{(g, z) \in [E, D] \times L(E) : (fg, z) \in F(E)\}.$$

With the following membership relation:

$$x(\in_C)F \iff (1_C, x) \in F(C)$$

Proof As previously stated, we will prove this result by showing that $\mathcal{P}L$, as defined above, has the universal property of a power-object for L.

Suppose we are given $B \stackrel{R}{\nrightarrow} L$ in $\mathbf{rel}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$. Define $B \xrightarrow{r} \mathcal{P}L$ by

$$r_C(x)(D) = \{(f, y) \in [D, C] \times L(D) : y(R_D)B(f)(x)\}$$

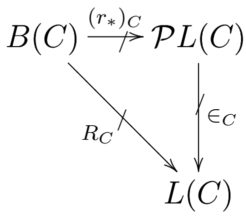
We first need to check r is natural. That is, given $D \xrightarrow{f} C$ in $\mathbf{C}$, we need

$$\begin{array}{ccc} B(C) & \xrightarrow{r_C} & \mathcal{P}L(C) \\ {\scriptstyle B(f)}\downarrow & & \downarrow{\scriptstyle PL(f)} \\ B(D) & \xrightarrow[r_D]{} & \mathcal{P}L(D) \end{array}$$

Let $E \in \mathbf{C}$, $g \in [E, D]$, and $z \in L(E)$. Then:

$$\frac{\dfrac{\dfrac{\dfrac{(g, z) \in \mathcal{P}L(f)r_c(x)(E)}{(fg, z) \in r_c(x)(E)}}{z[R_E]B(fg)(x)}}{z[R_E]B(g)(B(f)(x))}}{(g, z) \in r_D(B(f)(x))(E)}$$

Thus r is natural. We also need the following diagram to commute in **rel**(**set**):



Indeed, for $x \in B(C)$, $y \in L(C)$, we have:

$$\begin{array}{c} y(R_C)x \\ \hline y(R_C)1_C(x) \\ \hline (1_C, y) \in r_c(x)(C) \\ \hline y \in_C (r_*)_C(x) \end{array}$$

We have proven the existence of such an r. We now need to prove uniqueness. Suppose we have a map $B \xrightarrow{r} \mathcal{P}L$ such that the above triangle commutes. Then we have

$$\begin{array}{cl} (f, y) \in r_C(x)(D) & \\ \hline (1_D, y) \in \mathcal{P}L(f)(r_c(x))(D) & \text{(by definition of } \mathcal{P}L(F)) \\ \hline (1_D, y) \in r_D(B(f)(x))(D) & \text{(by naturality of r)} \\ \hline y \in_D r_D(B(f)(x) & \\ \hline y(R_D)B(f)(x) & \text{(since the triangle commutes)} \end{array}$$

Thus we have that such an r must be defined by

$$r_C(x)(D) = \{(f, y) \in [D, C] \times L(D) : y(R_D)B(f)(x)\}$$

as required. ■

3.3 Posets, the $\downarrow$ map, and the Down Object

An internal poset L in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ is simply a poset-valued functor.

Proposition 3.3.1 *$(L, \leq)$ is an internal poset in* $\mathbf{set}^{\mathbf{C}^{op}}$ $\Leftrightarrow$ *L takes values in* **ord**(**set**), *that is, $\forall C \in \mathbf{C}, (L(C), \leq (C))$ is a poset, and for all $D \xrightarrow{f} C$ in $\mathbf{C}$, $L(f)$ is an order-preserving function.*

Proof $(\Rightarrow)$ Since L is a poset, $\leq \hookrightarrow L \times L$ is reflexive and transitive. Thus for each $C \in \mathbf{C}$, $\leq(C) \hookrightarrow L(C) \times L(C)$ is also reflexive and transitive, hence $(L(C), \leq(C))$ is a poset. Moreover, the naturality square

$$\begin{array}{ccc} \leq(C) & \longrightarrow & L(C) \times L(C) \\ {\scriptstyle \leq(f)} \downarrow & & \downarrow {\scriptstyle L(f) \times L(f)} \\ \leq(D) & \longrightarrow & L(D) \times L(D) \end{array}$$

shows that each $L(f)$ is order-preserving.

$(\Leftarrow)$ If each $(L(C), \leq(C))$ is a poset, then we can define $\leq$ to be the functor determined by these $\leq(C)$'s. Then the reflexivity and transitivity of $\leq$ follows by the reflexivity and transitity of each $\leq(C)$. As noted above, the fact that each $L(f)$ is order-preserving is equivalent to the naturality of $\leq \hookrightarrow L \times L$. ■

To figure out what the down object is, we will first determine the $\downarrow$ arrow. We can do this without first knowing the down object since the down arrow is the name of the $\leq$ relation. Thus as long as we know the power object, we can determine the $\downarrow$ arrow.

Proposition 3.3.2 *For L an internal poset in $\mathbf{set}^{\mathbf{C}^{op}}$, the down-arrow map, $L \xrightarrow{\downarrow} \mathcal{P}L$ is defined by*

$$\downarrow_C(x)(D) = \{(f, y) : y \leq L(f)(x)\}$$

Proof We have the relation

$$L \overset{\leq}{\longrightarrow\!\!\!\!\!\!\!/\;\;} L$$

defined by, for $C \in \mathbf{C}, x_1, x_2 \in L(C)$,

$$(x_1 \leq_c x_2) \Leftrightarrow (x_1 \leq_{L(C)} x_2)$$

Then by the proof of Proposition 3.2.2, the name of the $\leq$ relation $\downarrow$, is

$$\downarrow_C(x)(D) = \{(f, y) : y \leq L(f)(x)\}$$

as required. ■

We are now in a position to determine what the down object is. We know that $\mathcal{D}L \hookrightarrow \mathcal{P}L$, so for each $C \in \mathbf{C}$, $\mathcal{D}L(C) \subseteq \mathcal{P}L(C)$. Hence the elements of $\mathcal{D}L(C)$ will be subfunctors of $\mathbf{C}(-, C) \times L$. We can now use the second property of the $\downarrow$ map: it is the Yoneda embedding for L (Proposition 2.2.3). Thus the Yoneda Lemma for L (Proposition 2.2.4) tells us that for $C \in \mathbf{C}$, $x \in L(C)$, $F \in \mathcal{D}L(C)$, we have

$$x \in_C F \Leftrightarrow \downarrow_C (x) \subseteq F$$

Now, ($\Leftarrow$) is always satisfied; however, ($\Rightarrow$) gives us all we need to know about the subfunctors that make up $\mathcal{D}L(C)$. Indeed, ($\Rightarrow$) means that if $x \in_C F$, and $y \leq L(f)(x)$, then $(f, y) \in F(D)$. But $x \in_C F$ implies $(1_C, x) \in F(C)$, so applying $F(f)$ to it, we get $(f, L(f)(x)) \in F(D)$. In other words, given that $(f, L(f)(x)) \in F(D)$ and $y \leq L(f)(x)$, $(f, y) \in F(D)$. In other words, we require that the set $\{y \in L(D) : (f, y) \in F(D)\}$ be down-closed.

Definition If $F \in \mathcal{P}L(C)$, call F *down-closed* if for each $D \in \mathbf{C}$, $f \in [D, C]$, the set $\{y \in L(D) : (f, y) \in F(D)\}$ is down-closed.

Using the above, we can now state our intended result:

Proposition 3.3.3 *For L a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, $\mathcal{D}L$ is the subfunctor of $\mathcal{P}L$ determined by*

$$\mathcal{D}L(C) = \{F \subseteq \mathbf{C}(-, C) \times L : F \textit{ is down-closed}\}$$

Proof Since $\mathcal{D}L \subseteq \mathcal{P}L$, we must have $\mathcal{D}L(f) = \mathcal{P}L(f)$ restricted to $\mathcal{D}L(C)$. We first need to check this restriction is valid. Suppose $w \leq z$, and $(g, z) \in \mathcal{D}L(f)(F)(E)$. Then

$$(gf, z) \in F(E) \Rightarrow (gf, w) \in F(E) \Rightarrow (g, w) \in \mathcal{D}L(f)(F)(E)$$

so indeed $\mathcal{P}L(f)$ restricts correctly to $\mathcal{D}L(f)$.

To prove $\mathcal{D}$ is the down object, we will check the universal property of Proposition 2.1.5. As in the proof of that proposition, we check that each of the neccesary restrictions is valid.

We first need to see if $\in$ restricts normally, ie., is $\mathcal{D}L \overset{\in}{\rightarrowtail\!\!\!\!\!\!/} L$ an order ideal? Suppose $z \leq x$, and $x(\in_C)F$. Then $(1_C, x) \in F(C)$, so $(1_C, y) \in F(C)$ (since $F(C)$ is down-closed), hence $y(\in_C)F$. If $x(\in_C)F \subseteq G$, then clearly $x \in_C G$. So $\in$ restricts correctly.

Finally, we need to check that given an order-ideal R, the name r factors through $\mathcal{D}L$, and is order-preserving. Suppose $(f, y) \in r_c(x)(D)$, and $y' \leq y$. Then we have $y' \leq y(R_D)L(f)(x)$, so $(f, y') \in r_c(x)(D)$, since R is an order ideal. Hence $r_C(x)$ is down-closed, so r factors through $\mathcal{D}L$.

Suppose $x_1 \leq x_2 \in L(C)$. We need $r_C(x_1) \subseteq r_C(x_2)$. Suppose $(f, y) \in r_C(x_1)(D)$. Then $y(R_D)L(f)(x_1) \leq L(f)(x_2)$, so $y(R_D)L(f)(x_2)$, and hence $(f, y) \in r_c(x_2)(D)$. Thus r is order-preserving. ■

3.4 The Evaluation Functors

The final detail we need to understand in the topos $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ are the evaluation functors. There is an evaluation functor

$$\begin{array}{rcl} \mathrm{ev}_C : \mathbf{set}^{\mathbf{C}^{\mathrm{op}}} & \longrightarrow & \mathbf{set} \\ F & \longmapsto & F(C) \end{array}$$

defined for each $C \in \mathbf{C}$. These functors allow us to better understand structures such as complete lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. The key property of the evaluation functors is the following:

Proposition 3.4.1 *For fixed $C \in \mathbf{C}$, the evaluation functor $\mathbf{set}^{\mathbf{C}^{op}} \xrightarrow{ev_C} \mathbf{set}$ is a partial geometric morphism, that is, it has a left adjoint which preserves pullbacks but not neccesarily terminal objects.*

Proof Define the functor

$$\begin{array}{rcl} \Phi : \mathbf{set} & \longrightarrow & \mathbf{set}^{\mathbf{C}^{\mathrm{op}}} \\ X & \longmapsto & \sum_{x \in X} \mathbf{C}(-, C) = X \cdot \mathbf{C}(-, C) \end{array}$$

We claim that $\Phi \dashv \mathrm{ev}_C$. Indeed:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi(X) \longrightarrow F}{\sum_{x\in X}\mathbf{C}(-,C) \longrightarrow F}}{\{\mathbf{C}(-,C) \xrightarrow{x} F : x \in X\}}}{X \longrightarrow [\mathbf{C}(-,C), F]}}{X \longrightarrow F(C)} \text{ by Yoneda}}{X \longrightarrow \mathrm{ev}_C(F)}$$

So we have found a left adjoint for ev_C. We now need to show that it preserves pullbacks. Let

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & \times & \downarrow \\ X_3 & \longrightarrow & X_4 \end{array}$$

be a pullback in **set**. We need to show that

$$\begin{array}{ccc} \sum_{X_1}\mathbf{C}(-,C) & \longrightarrow & \sum_{X_2}\mathbf{C}(-,C) \\ \downarrow & & \downarrow \\ \sum_{X_3}\mathbf{C}(-,C) & \longrightarrow & \sum_{X_4}\mathbf{C}(-,C) \end{array}$$

is a pullback in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. However, the above is a pullback if and only if, for each $D \in \mathbf{C}$,

$$\begin{array}{ccc} \sum_{X_1}\mathbf{C}(D,C) & \longrightarrow & \sum_{X_2}\mathbf{C}(D,C) \\ \downarrow & & \downarrow \\ \sum_{X_3}\mathbf{C}(D,C) & \longrightarrow & \sum_{X_4}\mathbf{C}(D,C) \end{array}$$

is a pullback in **set**. But this follows because of the fact that for any set S, $(-)\cdot S = \sum_{(-)} S$ preserves pullbacks. ∎

Chapter 4

CCD Theory in $\mathbf{set}^{\mathbf{C}^{op}}$

Now that we have both the general theory of CCD lattices and the particular details of the topos $\mathbf{set}^{\mathbf{C}^{op}}$, we can bring them together to try and understand the nature of CCD lattices in $\mathbf{set}^{\mathbf{C}^{op}}$.

4.1 Preservation of Lattice Structure

An important element of understanding CCD theory in $\mathbf{set}^{\mathbf{C}^{op}}$ is, for L a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, the functor $\mathcal{D}L$. In the first CCD paper [1], it was noted that for any $L \in \mathbf{ord}$, $\mathcal{D}L$ is not only a sup lattice, it is in fact always a CCD lattice. Thus any properties that a sup or CCD lattice in $\mathbf{set}^{\mathbf{C}^{op}}$ has, $\mathcal{D}L$ will have. Moreover, as we will shall see later, some of the properties which L may have can be obtained through the corresponding properties of $\mathcal{D}L$.

In fact, this technique is used in the third CCD paper [7], to try and determine which functors between toposes preserve CCD objects. Theorems 8, 9 and 11 of [7] together state the following:

Theorem 4.1.1 *Let* $\Gamma : \mathbf{E} \longrightarrow \mathbf{S}$ *be a left exact functor between toposes, and* $L \in \mathbf{ord}(\mathbf{E})$. *Let* $\mathcal{D}_{\mathbf{E}}$ *denote the down object mapping in* $\mathbf{E}$. *Then we have:*

1. *(L is a sup lattice $\Rightarrow \Gamma L$ is a sup lattice) $\Leftrightarrow \Gamma\mathcal{D}_{\mathbf{E}}L$ is a sup lattice.*

2. *If L has finite infima, (L is a locale $\Rightarrow \Gamma L$ is a locale) $\Leftrightarrow \Gamma\mathcal{D}_{\mathbf{E}}L$ is a locale.*

3. *If L is a sup lattice, (L is a CCD lattice $\Rightarrow \Gamma L$ is a CCD lattice) $\Leftrightarrow \Gamma\mathcal{D}_{\mathbf{E}}L$ is a CCD lattice.*

In other words, if Γ is left exact, Γ preserves (sup lattices, locales, CCD lattices) precisely when Γ of $\mathcal{D}L$ is a (sup lattice, locale, CCD lattice). This is exactly the method described above: to determine when an object is a (sup lattice, locale, CCD lattice), one determines what happens to $\mathcal{D}$ of that object.

Not only is the above theorem indicative of the method we will use, it is useful in itself. By Proposition 3.4.1, we know that the evaluation functor $L \mapsto L(C)$ is left exact. Thus:

Corollary 4.1.2 *For L a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, we have for each $C \in \mathbf{C}$:*

1. *(L is a sup lattice $\Rightarrow L(C)$ is a sup lattice) $\Leftrightarrow \mathcal{D}L(C)$ is a sup lattice.*
2. *If L has finite infima, (L is a locale $\Rightarrow L(C)$ is a locale) $\Leftrightarrow \mathcal{D}L(C)$ is a locale.*
3. *If L is a sup lattice, (L is a CCD lattice $\Rightarrow L(C)$ is a CCD lattice) $\Leftrightarrow \mathcal{D}L(C)$ is a CCD lattice.*

We will first apply this result to determine the nature of sup lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. As a point of notation, we will write $\mathcal{D}_S$ for the down functor in **set**, and reserve $\mathcal{D}$ for the down functor in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$.

4.2 Complete Lattices

By the result above, the first useful thing to determine is whether or not the sets $\mathcal{D}L(C)$ are sup lattices. In fact, this is the case:

Lemma 4.2.1 *If L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $C \in \mathbf{C}$, $\mathcal{D}L(C)$ is a sup lattice.*

Proof Since $\mathcal{D}L(C)$ is simply the set of down-closed subfunctors of $\mathbf{C}(-, C) \times L$, the order on $\mathcal{D}L(C)$ is inclusion. Thus the supremum operation on $\mathcal{D}L(C)$, if it exists, is simply union, that is, for an I-indexed family of down-closed subfunctors $\{F_i\}_{i \in I}$, and $D \in \mathbf{C}$,

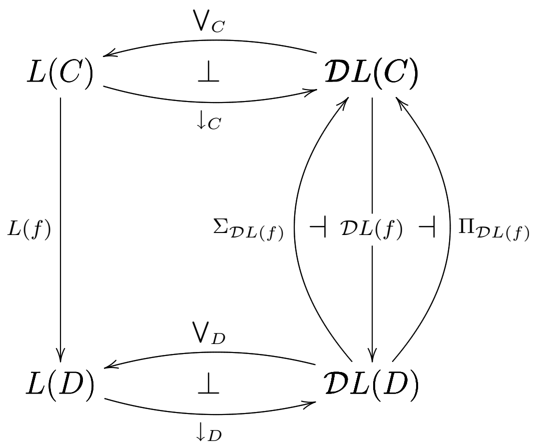
$$\left(\bigvee F_i\right)(D) = \bigcup [F_i(D)]$$

Thus the only thing to check is that this union actually defines an element of $\mathcal{D}L(C)$; in other words, we need this functor to be down-closed. But that is easy to see, for if $(f, y) \in \bigvee F_i(D)$, then $(f, y) \in F_j(D)$ for some j, and hence $z \leq y$ will imply $(f, z) \in F_j(D)$ and so in the union. ∎

Thus we have by Corollary 4.1.2:

Proposition 4.2.2 *If L is a sup lattice in* $\mathbf{set}^{\mathbf{C}^{op}}$*, then for each $C \in \mathbf{C}$, $L(C)$ is a sup lattice in* $\mathbf{set}$*.*

However, this is only one part of the requirement for L to be a sup lattice in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. We must also find conditions on the functions $L(f)$. The first thing we would like to show is that if L is a sup lattice, then the functions $L(f)$ have left and right adjoints $\Sigma_{L(f)}$ and $\Pi_{L(f)}$. Following our prescribed method, we will first show that for L any poset, $\mathcal{D}L$ has this property, ie. the functions $\mathcal{D}L(f)$ have left and right adjoints $\Sigma_{\mathcal{D}L(f)}$ and $\Pi_{\mathcal{D}L(f)}$. If L is also a sup lattice, then we would have the following diagram:



We could then try defining $\Sigma_{L(f)}$ to be $\bigvee_C \Sigma_{\mathcal{D}L(f)} \downarrow_D$. First, however, we need to prove the existence of $\Sigma_{\mathcal{D}L(f)}$ and $\Pi_{\mathcal{D}L(f)}$.

Lemma 4.2.3 *Suppose L is a poset in* $\mathbf{set}^{\mathbf{C}^{op}}$*. Then for all $D \xrightarrow{f} C$ in $\mathbf{C}$, the function $\mathcal{D}L(C) \xrightarrow{\mathcal{D}L(f)} \mathcal{D}L(D)$ has a left adjoint $\Sigma_{\mathcal{D}L(f)}$, defined for each $G \in \mathcal{D}L(D)$, $E \in \mathbf{C}$ by*

$$\Sigma_{\mathcal{D}L(f)}(G)(E) = \{(h, z) \in [E, C] \times L(E) : \exists g \in [E, D] : (g, z) \in G(E), h = fg\}$$

Proof First, we need to check that this actually defines a down-closed subfunctor of $\mathbf{C}(-, C) \times L$. Suppose we have a map $E_2 \xrightarrow{k} E_1$, and suppose $(h, z) \in \Sigma_{\mathcal{D}L(f)}(G)(E_1)$. We need $\Sigma_{\mathcal{D}L(f)}$ to have the same action as $\mathbf{C}(-, C) \times L$, so we need

$(hk, L(k)(z)) \in \Sigma_{\mathcal{D}L(f)}(G)(E_2)$. Now, since $(h, z) \in \Sigma_{\mathcal{D}L(f)}(G)(E_1)$, we have that $\exists g \in [E_1, D]$ such that $(g, z) \in G(E_1)$ and $h = fg$. So consider $gk \in [E_2, D]$. Then

$$\begin{aligned} & (g, z) \in G(E_1) \\ \Rightarrow \quad & G(k)(g, z) \in G(E_2) \\ \Rightarrow \quad & (gk, L(k)(z)) \in G(E_2) \end{aligned}$$

And $fgk = hk$ since $fg = h$. Thus the existence of gk demonstrates that $(h, z) \in \Sigma_{\mathcal{D}L(f)}(G)(E_2)$, so $\Sigma_{\mathcal{D}L(f)}(G)$ is indeed a subfunctor of $\mathbf{C}(-, C) \times L$.

Now we need it to be down-closed. Suppose $(h, z) \in \Sigma_{\mathcal{D}L(f)}(G)(E)$, and $w \leq z$. Then

$$\begin{aligned} & \exists g \in [E, D] \text{ s.t. } (g, z) \in G(E) \text{ and } h = fg \\ \Rightarrow \quad & (g, w) \in G(E) \text{ and } h = fg \text{ (since G is down-closed)} \\ \Rightarrow \quad & (h, w) \in \Sigma_{\mathcal{D}L(f)}(G)(E) \end{aligned}$$

So indeed $\Sigma_{\mathcal{D}L(f)}$ is down-closed.

Now we want $\Sigma_{\mathcal{D}L(f)} \dashv \mathcal{D}L(f)$. First, we need $G \subseteq \mathcal{D}L(f)\Sigma_{\mathcal{D}L(f)}(G)$. Suppose $(g, y) \in G(E)$. We need $(fg, y) \in \Sigma_{\mathcal{D}L(f)}(G)(E)$. Clearly the map g demonstrates that existence.

Next, we need $\Sigma_{\mathcal{D}L(f)}(\mathcal{D}L(f))(F) \subseteq F$. Suppose $(h, z) \in \Sigma_{\mathcal{D}L(f)}(\mathcal{D}L(f))(F)(E)$. Then

$$\begin{aligned} & \exists g \in [E, D] \text{ s.t. } (g, z) \in \mathcal{D}L(f)(F)(E) \text{ and } h = fg \\ \Rightarrow \quad & (fg, z) \in F(E) \text{ and } h = fg \\ \Rightarrow \quad & (h, z) \in F(E) \end{aligned}$$

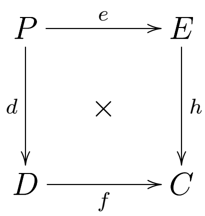
So we have the unit and co-unit, proving the adjunction. ■

To get the right adjoint to $\mathcal{D}L(f)$, we will assume $\mathbf{C}$ has pullbacks, an assumption we will use later when characterizing sup lattices in $\mathbf{set}^{\mathbf{C}^{\text{op}}}$.

Lemma 4.2.4 *Assume* $\mathbf{C}$ *has pullbacks, and suppose* L *is a poset in* $\mathbf{set}^{\mathbf{C}^{op}}$*. Then for* $D \xrightarrow{f} C$ *in* $\mathbf{C}$*, the function* $\mathcal{D}L(C) \xrightarrow{\mathcal{D}L(f)} \mathcal{D}L(D)$ *has a right adjoint* $\Pi_{\mathcal{D}L(f)}$*, defined for each* $G \in \mathcal{D}L(D)$*,* $E \in \mathbf{C}$ *by*

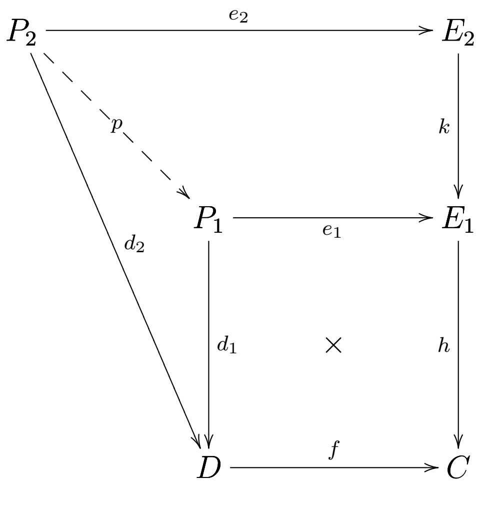
$$\Pi_{\mathcal{D}L(f)}(G)(E) = \{(h, z) \in [E, C] \times L(E) : (d, L(e)(z)) \in G(P)\}$$

where (P, d, e) *is the pullback of* h *and* f *in* $\mathbf{C}$*:*



Proof Again, we first need to check that this actually defines an element of $\mathcal{D}L(C)$. Suppose that $(h, z) \in \Pi_{\mathcal{D}L(f)}(G)(E_1)$, and we have an arrow $E_2 \xrightarrow{k} E_1$ in $\mathbf{C}$. We need $(hk, L(k)(z)) \in \Pi_{\mathcal{D}L(f)}(G)(E_2)$.

Let (P_1, d_1, e_1) be the pullback of (f, h), and (P_2, d_2, e_2) the pullback of (f, hk). Then we have:



p exists due to the pullback property of (P_1, d_1, e_1). Then:

$$\begin{aligned} & (h, z) \in \Pi_{\mathcal{D}L(f)}(G)(E_1) \\ \Rightarrow\ & (d_1, L(e_1)(z)) \in G(P_1) \\ \Rightarrow\ & G(p)(d_1, L(e_1)(z)) \in G(P_2) \\ \Rightarrow\ & (d_1 p, L(p)L(e_1)(z)) \in G(P_2) \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad (d_2, L(e_2)L(k)(z)) \in G(P_2) \text{ (by the commutivity of the above)} \\
&\Rightarrow \quad (hk, L(k)(z) \in \Pi_{\mathcal{D}L(f)}(G)(E_2) \text{ as required.}
\end{aligned}$$

We also need the functor $\Pi_{\mathcal{D}L(f)}(G)$ to be down-closed. Suppose $(h, z) \in \Pi_{\mathcal{D}L(f)}(G)(E)$, and $w \leq z$. Then $L(e)(w) \leq L(e)(z)$, so:

$$\begin{aligned}
& \quad (h, z) \in \Pi_{\mathcal{D}L(f)}(G)(E) \\
&\Rightarrow \quad (d, L(e)(z)) \in G(P) \\
&\Rightarrow \quad (d, L(e)(w)) \in G(P) \text{ (since G is down-closed)} \\
&\Rightarrow \quad (h, w) \in \Pi_{\mathcal{D}L(f)}(G)(E)
\end{aligned}$$

Now we want $\mathcal{D}L(f) \dashv \Pi_{\mathcal{D}L(f)}$. We first need $F \subseteq \Pi_{\mathcal{D}L(f)}\mathcal{D}L(f)(F)$. Now, note that:

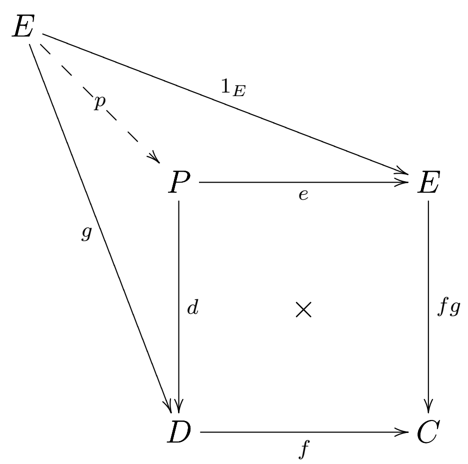
$$\begin{array}{c}
(h, z) \in \Pi_{\mathcal{D}L(f)}\mathcal{D}L(f)(F)(E) \\
\hline
(d, L(e)(z)) \in \mathcal{D}L(f)(F)(P) \\
\hline
(fd, L(e)(z)) \in F(P) \\
\hline
(he, L(e)(z) \in F(P)
\end{array} \text{ (by the pullback square)}$$

But applying $F(e)$ to $(h, z) \in F(E)$ gives $(he, L(e)(z)) \in F(P)$.

We also need $\mathcal{D}L(f)\Pi_{\mathcal{D}L(f)}(G) \subseteq G$. Indeed:

$$\frac{(g, z) \in \mathcal{D}L(f)\Pi_{\mathcal{D}L(f)}(G)(E)}{(fg, z) \in \Pi_{\mathcal{D}L(f)}(G)(E)}$$

Let (P, d, e) be the pullback of f and fg. Since $(E, 1_E, g)$ also makes the square commute, we have a map $p \in [E, P]$ as follows:



Then:

$$\begin{aligned} & (fg, z) \in \Pi_{\mathcal{D}L(f)}(G)(E) \\ \Rightarrow\ & (d, L(e)(z)) \in G(P) \text{ by definition of } \Pi_{\mathcal{D}L(f)} \\ \Rightarrow\ & G(p)(d, L(e)(z)) \in G(E) \\ \Rightarrow\ & (dp, L(p)L(e)(z)) \in G(E) \\ \Rightarrow\ & (g, z) \in G(E) \text{ by commutivity of above} \end{aligned}$$

Thus $\mathcal{D}L(f)\Pi_{\mathcal{D}L(f)}(G) \subseteq G$, as required. ■

Now that we have proven the existence of the left and right adjoints to $\mathcal{D}L(f)$, we can try the definition of $\Sigma_{L(f)}$ mentioned above. From now on, for ease of notation, we will simply write Σ_f for $\Sigma_{L(f)}$ and Π_f for $\Pi_{L(f)}$

Proposition 4.2.5 *If L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, then for any $D \xrightarrow{f} C$, L(f) has a left adjoint Σ_f, defined by*

$$\Sigma_f = \bigvee_C \Sigma_{\mathcal{D}L(f)} \downarrow_D$$

Proof Consider, for $x \in L(C)$, $y \in L(D)$,

$$\cfrac{\cfrac{\cfrac{\cfrac{\bigvee_C \Sigma_{\mathcal{D}L(f)} \downarrow_D (y) \leq x}{\Sigma_{\mathcal{D}L(f)} \downarrow_D (y) \leq \downarrow_C (x)} \text{(by lemma 4.2.3)}}{\downarrow_D (y) \leq DL(f) \downarrow_C (x)} \text{(by naturality)}}{\downarrow_D (y) \leq \downarrow_D L(f)(x)} \text{(since } \downarrow \text{ is Yoneda)}}{y \leq L(f)(x)}$$

Thus $\bigvee_C \Sigma_{\mathcal{D}L(f)} \downarrow_D \dashv L(f)$. ■

For the right adjoint Π_f, one uses $\mathcal{U}L$ (see Section 2.6) instead of $\mathcal{D}L$. It is easy to check that $\mathcal{U}L(C) \xrightarrow{\mathcal{U}L(f)} \mathcal{U}L(D)$ has the same action on subfunctors as $\mathcal{D}L(f)$. Then we will define $\mathcal{U}L(D) \xrightarrow{\Sigma_{\mathcal{U}L(f)}} \mathcal{U}L(C)$ to have the same action as $\Sigma_{\mathcal{D}L(f)}$. Since we order $\mathcal{U}L$ by reverse inclusion, it is also easy to check that we have $\mathcal{U}L(f) \dashv \Sigma_{\mathcal{U}L(f)}$. We can then use these maps to define Π_f, the right adjoint to $L(f)$.

Proposition 4.2.6 *If L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, then for any $D \xrightarrow{f} C$, $L(f)$ has a right adjoint Π_f, defined by*

$$\Pi_f = \bigwedge_C \Sigma_{\mathcal{U}L(f)} \uparrow_D$$

Proof Exactly as for the proof of Proposition 4.2.5, since all adjunctions go in the opposite direction. ■

However, more can be said about the adjunction $\Sigma_f \dashv L(f)$: it also satisfies the Beck-Chevalley conditions. Again, we will first show that the adjunction $\Sigma_{\mathcal{D}L(f)} \dashv \mathcal{D}L(f)$ has this property, then use the fact that we constructed Σ_f out of $\Sigma_{\mathcal{D}L(f)}$ to show that $\Sigma_f \dashv L(f)$ satisfies Beck-Chevalley.

Lemma 4.2.7 *If L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $D \xrightarrow{f} C$, the adjunction $\Sigma_{\mathcal{D}L(f)} \dashv \mathcal{D}L(f)$ satisfies the Beck-Chevalley conditions.*

Proof We need

$$\begin{array}{ccc} P & \xrightarrow{e} & E \\ {\scriptstyle d}\downarrow & \times & \downarrow{\scriptstyle h} \\ D & \xrightarrow[f]{} & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{D}L(E) & \xrightarrow{\mathcal{D}L(e)} & \mathcal{D}L(P) \\ {\scriptstyle \Sigma_{\mathcal{D}L(h)}}\downarrow & = & \downarrow{\scriptstyle \Sigma_{\mathcal{D}L(d)}} \\ \mathcal{D}L(C) & \xrightarrow[\mathcal{D}L(f)]{} & \mathcal{D}L(D) \end{array}$$

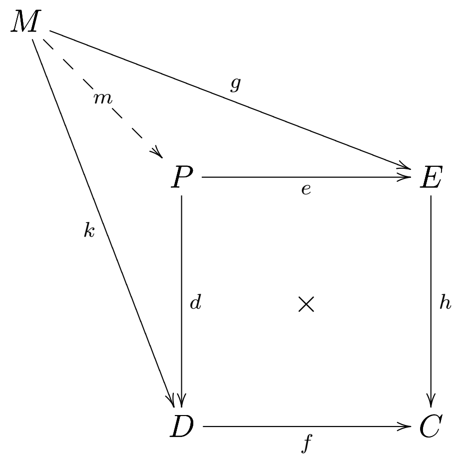
where the pullback is taken in $\mathbf{C}$. So we need $\forall H \in \mathcal{D}L(E), M \in \mathbf{C}$,

$$\mathcal{D}L(f)\Sigma_{\mathcal{D}L(h)}(H)(M) = \Sigma_{\mathcal{D}L(d)}\mathcal{D}L(e)(H)(M)$$

Let LS stand for the left side of the equation, and RS for the right side. Then:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{(k,w) \in \text{LS}}{(fk,w) \in \Sigma_{\mathcal{D}L(h)}(H)(M)}}{\exists g \in [M,E] : (g,w) \in H(M), fk = hg}}{\exists m \in [M,P] : (em,w) \in H(M), k = dm}\;(*)}{\exists m \in [M,P] : (m,w) \in \mathcal{D}L(e)H(M), k = dm}}{(k,w) \in \text{RS}}$$

where ($\Downarrow$) for * follows by the pullback property:



and ($\Uparrow$) for * follows by letting $g = em$. ■

Proposition 4.2.8 *If L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $D \xrightarrow{f} C$, the adjunction $\Sigma_f \dashv L(f)$ satisfies the Beck-Chevalley conditions.*

Proof Given the same pullback in the proof of Lemma 4.2.7, we have:

$$\begin{aligned}
L(f)\Sigma_h(w) &= L(f)\bigvee_C \Sigma_{\mathcal{D}L(h)} \downarrow_E (w) \text{ (Proposition 4.2.3)}\\
&= \bigvee_D \mathcal{D}L(f)\Sigma_{\mathcal{D}L(h)} \downarrow_E (w) \text{ (naturality)}\\
&= \bigvee_D \Sigma_{\mathcal{D}L(d)}\mathcal{D}L(e) \downarrow_E (w) \text{ (Lemma 4.2.7)}\\
&= \bigvee_D \Sigma_{\mathcal{D}L(d)} \downarrow_P L(e)(w) \text{ (naturality)}\\
&= \Sigma_d L(e)(w) \text{ (Proposition 4.2.3)}
\end{aligned}$$

as required. ■

Together, all these conditions imply that L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. However, the proof requires that the category $\mathbf{C}$ have pullbacks, so as to take advantage of the Beck-Chevalley condition.

Theorem 4.2.9 *When the category $\mathbf{C}$ has pullbacks, L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$ if and only if*

1. *For each $C \in \mathbf{C}$, $L(C)$ is a sup lattice.*

2. *For each $D \xrightarrow{f} C$ in $\mathbf{C}$, $L(f)$ has left and right adjoints Σ_f and Π_f.*

3. *The adjunctions $\Sigma_f \dashv L(f)$ satisfy Beck-Chevalley.*

Proof ($\Rightarrow$) by Propositions 4.2.2, 4.2.5, 4.2.6, and 4.2.8.

($\Leftarrow$) Define $\bigvee : \mathcal{D}L \longrightarrow L$ by

$$\bigvee_C F = \bigvee_{D \in \mathbf{C}} \; \bigvee_{(f,y) \in F(D)} \Sigma_f(y)$$

We first need to check that this is natural, ie. we need

$$\begin{CD} E \\ @VgVV \\ D \end{CD} \quad \Rightarrow \quad \begin{CD} \mathcal{D}L(D) @>{\bigvee_D}>> L(D) \\ @V{\mathcal{D}L(g)}VV @VV{L(g)}V \\ \mathcal{D}L(E) @>>{\bigvee_E}> L(E) \end{CD}$$

So we need to show that for each $F \in \mathcal{D}L(D)$,

$$L(g) \bigvee_D (F) \leq \bigvee_E \mathcal{D}L(g)(F)$$

and vice versa. Now,

$$L(g) \bigvee_D (F) = \bigvee_{B \in \mathbf{C}} \; \bigvee_{(f,y) \in F(B)} L(g)\Sigma_f(y)$$

since $L(g)$ has a right adjoint, and hence preserves $\bigvee$. Consider now the pullback of f and g,

$$\begin{CD} A @>b>> B \\ @VeVV \times @VVfV \\ E @>>g> D \end{CD}$$

which gives the Beck-Chevalley square:

$$\begin{array}{ccc} L(B) & \xrightarrow{L(b)} & L(A) \\ \Sigma_f \downarrow & = & \downarrow \Sigma_e \\ L(D) & \xrightarrow[L(g)]{} & L(E) \end{array}$$

So $L(g)\Sigma_f(y) = \Sigma_e L(b)(y)$. Consider the pair $(e, L(b)(y))$. We claim it is in $\mathcal{D}L(g)(F)(A)$. Indeed,

$$(ge, L(b)(y)) = (fb, L(b)(y) = F(b)(f, y)$$

So since $(f, y) \in \mathcal{D}L(g)(F)(A)$, $(ge, L(b)(y)) \in F(A)$, so $(e, L(b)(y))$ is in $\mathcal{D}L(g)(F)(A)$. So

$$L(g) \bigvee_D F \leq \bigvee_{A \in \mathbf{C}} \bigvee_{(e,z) \in \mathcal{D}L(g)(F)(A)} \Sigma_e(z) = \bigvee_E \mathcal{D}L(g)F$$

We will now demonstrate the opposite inequality. Suppose that we have $(e, z) \in \mathcal{D}L(g)(F)(A)$. Since $\Sigma_g \dashv L(g)$, we have

$$\begin{aligned} [\Sigma_e(z)] &\leq L(g)\Sigma_g[\Sigma_e(z)] \\ &\leq L(g)\Sigma_{ge}(z) \text{ (since adjoints compose)} \end{aligned}$$

Now, since $(e, z) \in \mathcal{D}L(g)(F)(A)$, $(ge, z) \in F(A)$. So $L(g)\Sigma_{ge}(z)$ appears in the sup of $L(g) \bigvee_D F$, and hence

$$\bigvee_{A \in \mathbf{C}} \bigvee_{(e,z) \in \mathcal{D}L(g)(F)(A)} \Sigma_e(z) \leq L(g) \bigvee_D F$$

Finally, we need to check that we have $\bigvee \dashv \downarrow$. That is, we need the unit $\bigvee_C \downarrow_C (x) \leq x$ and the co-unit $F \subseteq \downarrow_C \bigvee_C(F)$.

For the unit, suppose $(f, y) \in \downarrow_C (x)(D)$. Then $y \leq L(f)(x)$, so $\Sigma_f(y) \leq x$. Thus $\bigvee_C \downarrow_C (x) = \bigvee_{D \in \mathbf{C}} \bigvee_{(f,y) \in \downarrow_C (x)(D)} \Sigma_f(y) \leq x$. Note that x appears in the sup of $\downarrow_C (x)$, so we also have $x \leq \bigvee_C \downarrow_C (x)$. Thus $\bigvee_C \downarrow_C (x) = x$, a fact we will make use of later.

For the co-unit, since both F and $\downarrow_C \bigvee_C(F)$ are subfunctors of $\mathbf{C}(-, C) \times L$, it suffices to prove that $\forall D \in \mathbf{C}$, $F(D) \subseteq \downarrow_C \bigvee_C(F)(D)$. Suppose $(f, y) \in F(D)$. Then $\Sigma_f(y) \leq \bigvee_C(F)$, so $y \leq L(f)(\bigvee_C(F))$, and thus $(f, y) \in \downarrow_C [\bigvee_C(F)](D)$. ■

4.3 Locales

As before, to understand the nature of a locale in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, the first thing we need to check is that the ordered sets $\mathcal{D}L(C)$ are locales.

Lemma 4.3.1 *If L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $C \in \mathbf{C}$, $\mathcal{D}L(C)$ is a locale.*

Proof Let $(F_i)_{i\in I}$ and $(G_j)_{j\in J}$ be two elements of $\mathcal{D}_S(\mathcal{D}L(C))$. Then for $D \in \mathbf{C}$,

$$\frac{\dfrac{(f,y) \in [(\bigvee F_i) \wedge (\bigvee G_j)]\,(D)}{\exists i \in I : (f,y) \in F_i(D), \exists j \in J : (f,y) \in G_j(D)}}{(f,y) \in [\bigvee((F_i)_{i\in I} \wedge (G_j)_{j\in J})]\,(D)}$$

Thus $(\bigvee F_i) \wedge (\bigvee G_j) = \bigvee((F_i)_{i\in I} \wedge (G_j)_{j\in J})$, so $\mathcal{D}L(C)$ is a locale. ■

Then by corollary 4.1.2, we have:

Proposition 4.3.2 *If L is a locale in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $C \in \mathbf{C}$, $L(C)$ is a locale.*

The additional property that a locale will have in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ is the existence of Frobenius reciprocity on the adjunctions $\Sigma_f \dashv L(f)$.

Definition Suppose X and A are sup lattices, with **ord** maps $X \xrightarrow{f} A$, $A \xrightarrow{g} X$, and $g \dashv f$. Then say that the adjunction $g \dashv f$ satisfies *Frobenius reciprocity* if $\forall x \in X, a \in A$,

$$g(f(x) \wedge a) = x \wedge g(a)$$

As in the previous section, we will first look at the existence of this property on $\mathcal{D}L$.

Lemma 4.3.3 *If L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $D \xrightarrow{f} C$, the adjunction $\Sigma_{\mathcal{D}L(f)} \dashv \mathcal{D}L(f)$ satisfies Frobenius reciprocity.*

Proof We need, for all $D \xrightarrow{f} C$, $F \in \mathcal{D}L(C), G \in \mathcal{D}L(D)$,

$$\Sigma_{\mathcal{D}L(f)}[\mathcal{D}L(f)(F) \wedge G] = F \wedge \Sigma_{\mathcal{D}L(f)}G$$

Indeed, for $E \in \mathbf{C}$:

$$\frac{\frac{\frac{\frac{(h,z)\in \mathrm{LS}(E)}{\exists g\in[E,D]:(g,z)\in\mathcal{D}L(f)(F)(E),(g,z)\in G(E),h=fg}}{\exists g\in[E,D]:(fg,z)\in F(E),(g,z)\in G(E),h=fg}}{(h,z)\in F(E),(h,z)\in\Sigma_{\mathcal{D}L(f)}G(E)}}{(h,z)\in \mathrm{RS}(E)}$$

■

Proposition 4.3.4 *If L is a locale in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $D \xrightarrow{f} C$, $\Sigma_f \dashv L(f)$ satisfies Frobenius reciprocity.*

Proof Consider:

$$\begin{aligned}
\Sigma_f(L(f)(x)\wedge y) &= \bigvee_C \Sigma_{\mathcal{D}L(f)} \downarrow_D (L(f)(x)\wedge y) \text{ by Proposition 4.2.5}\\
&= \bigvee_C \Sigma_{\mathcal{D}L(f)}(\downarrow_D L(f)(x)\wedge \downarrow_D (y)) \text{ since L is a locale}\\
&= \bigvee_C \Sigma_{\mathcal{D}L(f)}(\mathcal{D}L(f) \downarrow_C (x)\wedge \downarrow_D (y)) \text{ naturality}\\
&= \bigvee_C (\downarrow_C (x)\wedge \Sigma_{\mathcal{D}L(f)} \downarrow_D (y)) \text{ by Lemma 4.3.3}\\
&= \bigvee_C \downarrow_C (x)\wedge \bigvee_C \Sigma_{\mathcal{D}L(f)} \downarrow_D (y) \text{ since } L \text{ is a locale}\\
&= x\wedge \Sigma_f(y)
\end{aligned}$$

■

These are precisely the conditions we need for L to be a locale.

Theorem 4.3.5 *When $\mathbf{C}$ has pullbacks, and L a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, L is also a locale if and only if*

1. *For each $C \in \mathbf{C}$, the sets $L(C)$ are locales.*
2. *For each $D \xrightarrow{f} C$ in $\mathbf{C}$, the adjunction $\Sigma_f \dashv L(f)$ satisfies Frobenius reciprocity.*

Proof (⇒) by Propositions 4.3.2 and 4.3.4.

($\Leftarrow$) Let F_1, F_2 be elements of $\mathcal{D}L(C)$. We need to show that

$$\bigvee_C F_1 \wedge \bigvee_C F_2 = \bigvee_C (F_1 \cap F_2)$$

First, note that $F_1 \cap F_2 \subseteq F_1$, so $\bigvee_C (F_1 \cap F_2) \leq \bigvee_C F_1$, and similarly $\bigvee_C (F_1 \cap F_2) \leq \bigvee_C F_2$. Hence $\bigvee_C (F_1 \cap F_2) \leq \bigvee_C F_1 \wedge \bigvee_C F_2$.

For the other direction:

$$\begin{aligned}
& \bigvee_C F_1 \wedge \bigvee_C F_2 \\
= \ & \bigvee_C F_1 \wedge \left(\bigvee_{L(C)} \{\Sigma_{f_2}(y_2) : (f_2, y_2) \in F_2(D_2), D_2 \in \mathbf{C}\} \right) \\
= \ & \bigvee_{L(C)} \left\{ \bigvee_C \{F_1 \wedge \Sigma_{f_2}(y_2) : (f_2, y_2) \in F_2(D_2), D_2 \in \mathbf{C}\} \right\} \text{ since } L(C) \text{ is a locale} \\
= \ & \bigvee_{L(C)} \left\{ \Sigma_{f_2}[L(f_2)(\bigvee_C F_1) \wedge (y_2)] : (f_2, y_2) \in F_2(D_2), D_2 \in \mathbf{C} \right\} \text{ by Frobenius} \\
= \ & \bigvee_{L(C)} \left\{ \Sigma_{f_2} \bigvee_{L(D_2)} \{L(f_2)\Sigma_{f_1}(y_1) \wedge (y_2) : (f_1, y_1) \in F_1(D_1)\} : (f_2, y_2) \in F_2(D_2) \right\}
\end{aligned}$$

With the last equality following since $F(D_2)$ is a locale. We now take the pullback of each f_1, f_2 and apply Beck-Chevalley.

$$\begin{array}{ccc}
P & \xrightarrow{d_1} & D_1 \\
{\scriptstyle d_2}\downarrow & \times & \downarrow{\scriptstyle f_1} \\
D_2 & \xrightarrow[f_2]{} & C
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
L(D_1) & \xrightarrow{L(d_1)} & L(P) \\
{\scriptstyle \Sigma_{f_1}}\downarrow & & \downarrow{\scriptstyle \Sigma_{d_2}} \\
L(C) & \xrightarrow[L(f_2)]{} & L(D_2)
\end{array}$$

Then the final term of the above equation becomes

$$\begin{aligned}
& \bigvee_{L(C)} \left\{ \Sigma_{f_2} \bigvee_{L(D_2)} \{\Sigma_{d_2} L(d_1)(y_1) \wedge (y_2)\} : (f_1, y_1) \in F_1(D_1), (f_2, y_2) \in F_2(D_2) \right\} \\
= \ & \bigvee_{L(C)} \left\{ \Sigma_{f_2} \bigvee_{L(D_2)} \{\Sigma_{d_2}(L(d_1)(y_1) \wedge L(d_2)(y_2))\} : (f_1, y_1) \in F_1(D_1), (f_2, y_2) \in F_2(D_2) \right\} \\
= \ & \bigvee_{L(C)} \{\Sigma_{f_2 d_2}(L(d_1)(y_1) \wedge L(d_2)(y_2)) : (f_1, y_1) \in F_1(D_1), (f_2, y_2) \in F_2(D_2)\} \ \ (*)
\end{aligned}$$

With the last equality by Frobenius. We now claim that $(f_2d_2, L(d_1)(y_1) \wedge L(d_2)(y_2)) \in F_1(P) \cap F_2(P)$. Indeed:

$$\begin{array}{ll} & (f_2, y_2) \in F_2(D_2) \\ \Rightarrow & F(d_2)(f_2, y_2) \in F_2(P) \\ \Rightarrow & (f_2d_2, L(d_2)(y_2)) \in F_2(P) \\ \Rightarrow & (f_2d_2, L(d_2)(y_2) \wedge L(d_1)(y_1) \in F_2(P)) \end{array}$$

Similarly, $(f_2d_2, L(d_2)(y_2) \wedge L(d_1)(y_1)) \in F_2(P)$ by using the fact that $f_2d_2 = f_1d_1$. Thus each element of (*) is in $\bigvee_C(F_1 \cap F_2)$, and therefore we have the other inequality $\bigvee_C F_1 \wedge \bigvee_C F_2 \leq \bigvee_C(F_1 \cap F_2)$. ■

4.4 CCD Lattices

The first thing we need to understand is the $\ll$ relation in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. As noted in Section 2.4, $\ll$ relation arises as the right Kan extension of $\downarrow^*$ along $\bigvee_*$. For $D \in \mathbf{C}$, we obtain the following diagram in $\mathbf{rel}(\mathbf{set})$:

$$\begin{array}{ccc} \mathcal{D}_S L(D) & \xrightarrow{(\bigvee_*)} & L(D) \\ & {\scriptstyle (\downarrow^*)} \searrow & \downarrow {\scriptstyle \ll_D} \\ & & L(D) \end{array}$$

So by the diagram, for $y_1, y_2 \in L(D)$,

$$(y_1 \ll_D y_2) \Longleftrightarrow \left(\forall G \in \mathcal{D}L(D), y_2 \leq \bigvee_D (G) \Rightarrow \downarrow_D (y_1) \subseteq G \right)$$

Then the universal property of $\mathcal{D}$ (Proposition 2.1.5), gives us the $\Downarrow$ map, stated here as a proposition:

Proposition 4.4.1 *Let L be a complete lattice in $\mathbf{set}^{\mathbf{C}^{op}}$. Then the map $\Downarrow: L \longrightarrow \mathcal{D}L$ is defined by:*

$$\begin{array}{rcl} \Downarrow_C (x)(D) & = & \{(f, y) \in [D, C] \times L(D) : y \ll_D L(f)(x)\} \\ & = & \{(f, y) : L(f)(x) \leq \bigvee_D G \ \text{and} \ \ z \leq L(g)(y) \Rightarrow (g, z) \in G(E)\} \end{array}$$

As before, to understand CCD lattices in $\mathbf{set}^{\mathbf{C}^{op}}$, the first thing we need to check is that the sets $\mathcal{D}L(C)$ are CCD lattices.

Lemma 4.4.2 *If L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, then for each $C \in \mathbf{C}$, $\mathcal{D}L(C)$ is a CCD lattice.*

Proof We know $\mathcal{D}L(C)$ is a sup lattice, with $\bigvee F_i(D) = \bigcup F_i(D)$. So we have for the lattice $\mathcal{D}L(C)$ a $\Downarrow$ map, defined by $\Downarrow(F) = \{G : G \ll F\}$, where

$$G \ll F \Leftrightarrow \text{ for } (F_i)_{i\in I} \in \mathcal{D}L(C), F \subseteq \bigvee\{F_i : i \in I\} \Rightarrow G \in (F_i)_{i\in I}$$

By definition, we have the co-unit $\Downarrow(\bigvee\{G_i\}_{i\in I}) \subseteq \{G_i\}_{i\in I}$, so all we need is $F \subseteq \bigvee \Downarrow(F)$.

Suppose $(f, y) \in F(D)$. We need a $G \ll F$ such that $(f, y) \in G(D)$. Define

$$G = \Sigma_{\mathcal{D}L(f)}(\downarrow_D (y))$$

Now $(1_D, y) \in \downarrow_D (y)$, so the arrow 1_D proves that $(f, y) \in G(D)$.

So we only need to show that $G \ll F$. Suppose $F \subseteq \bigvee\{F_i : i \in I\}$. Then $\exists i \in I$ such that $(f, y) \in F_i(D)$. We claim that $G \subseteq F_i$. Suppose that $(h, z) \in G(E)$. Then $\exists g \in [E, D]$ such that $z \leq L(g)(y)$ and $h = fg$. Then

$$\begin{aligned}
& (f, y) \in F_i(D) \\
\Rightarrow\ & F_i(g)(f, y) \in F_i(E) \\
\Rightarrow\ & (fg, L(g)(y)) \in F_i(E) \\
\Rightarrow\ & (h, L(g)(y)) \in F_i(E) \\
\Rightarrow\ & (h, z) \in F_i(E) \text{ (since } F_i \text{ is down-closed)}
\end{aligned}$$

Thus $G \subseteq F_i$, so $G \in (F_i)_{i\in I}$, and thus $G \ll F$. Hence the unit of the adjunction exists, and $\mathcal{D}L(C)$ is a CCD lattice. ■

Thus we have by Corollary 4.1.2:

Proposition 4.4.3 *If L is a CCD lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, then each set $L(C)$ is CCD.*

Now, as with sup lattices and locales, we need to find conditions on the functions $L(f)$, that together with the assumption that each $L(C)$ is CCD, will imply that L is CCD. Thinking about Theorem 2.3.4, one might suspect that since L a sup lattice implied $L(f)$ had a left adjoint Σ_f, L a CCD lattice might imply that Σ_f itself has a left adjoint.

Unfortunately, this is not the case for most categories **C**. In fact, one doesn't even have Σ_f preserving finite meets. To justify this, we will show that $\Sigma_{\mathcal{D}L(f)}$ does not preserve finite meets. Suppose $G_1, G_2 \in \mathcal{D}L(D)$. Preserving finite meets would mean that $\Sigma_{\mathcal{D}L(f)}(G_1 \cap G_2) = \Sigma_{\mathcal{D}L(f)}(G_1) \cap \Sigma_{\mathcal{D}L(f)}(G_2)$. Consider:

$$\frac{(h,z) \in \Sigma_{\mathcal{D}L(f)}(G_1 \cap G_2)(E)}{\exists g \in [E,D] : (g,z) \in G_1(E), (g,z) \in G_2(E), h = fg}$$

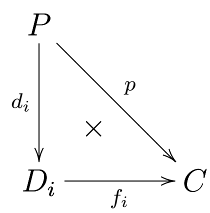
While:

$$\frac{(h,z) \in \Sigma_{\mathcal{D}L(f)}(G_1) \cap \Sigma_{\mathcal{D}L(f)}(G_2)(E)}{\exists g_1, g_2 \in [E,D] : (g_1,z) \in G_1(E), (g_2,z) \in G_2(E), h = fg_1 = fg_2}$$

One can see that the first implies the second by taking $g_1 = g = g_2$, but we can not go in the opposite direction unless we know that the maps g_1 and g_2 are equal. Obviously, we can not assume this unless the category is something specific such as $\mathbf{C} = \mathbf{2}$.

So the condition must be a bit a more complex than simply asking for a left adjoint to Σ_f. This condition we are looking for turns out to be something quite interesting. It is a combination of the Beck-Chevalley and Frobenius reciprocity conditions, but uses wide pullbacks instead of just finite pullbacks.

Definition Suppose L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Say that L satisfies *wide Frobenius reciprocity* if for any set of maps in **C**, $(D_i \xrightarrow{f_i} C)_{i \in I}$, with wide pullback:



together with any family $(y_i \in L(D_i))_{i \in I}$, the following equality is satisfied:

$$\bigwedge_{L(C)} \{\Sigma_{f_i}(y_i) : i \in I\} = \sum_p \left(\bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\} \right)$$

Though it may not appear so at first glance, Frobenius reciprocity is a special case of wide Frobenus reciprocity. Recall that Frobenius reciprocity says

$$\Sigma_f(L(f)(x) \wedge y) = x \wedge \Sigma_f(y)$$

To show this given wide Frobenius reciprocity, take $f_1 = 1_C$, $f_2 = f$, $y_1 = x$, and $y_2 = y$. Then the pullback of f_1 and f_2 is

$$\begin{array}{ccc} D & \xrightarrow{1_D} & D \\ {\scriptstyle f}\downarrow & \times & \downarrow{\scriptstyle f} \\ C & \xrightarrow[1_C]{} & C \end{array}$$

and so wide Frobenius reciprocity says

$$\Sigma_{f_1}(y_1) \wedge \Sigma_{f_2}(y_2) = \Sigma_p(L(d_1)(y_1) \wedge L(d_2)(y_2))$$

that is,

$$x \wedge \Sigma_f(y) = \Sigma_f(L(f)(x) \wedge y)$$

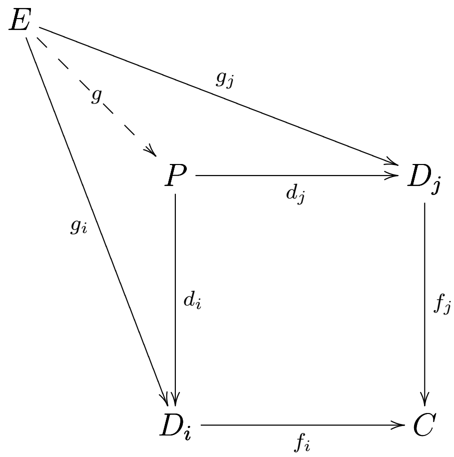
As usual, we first show that $\mathcal{D}L$ satisfies wide Frobenius reciprocity.

Lemma 4.4.4 *For L a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, $\mathcal{D}L$ satisfies wide Frobenius reciprocity.*

Proof We need, for any family $(G_i \in \mathcal{D}L(D_i))_{i \in I}$,

$$\bigcap_{L(C)} \{\Sigma_{\mathcal{D}L(f_i)}(G_i) : i \in I\} = \Sigma_{\mathcal{D}L(p)} \left(\bigcap_{\mathcal{D}L(P)} \{\mathcal{D}L(d_i)(G_i) : i \in I\} \right)$$

($\subseteq$) Suppose $(h, z) \in LS(E)$. That is, for each $i \in I$, $\exists D_i \xrightarrow{g_i} C$ such that $(g_i, z) \in G_i(E)$, and $h = f_i g_i$. Thus by the pullback, we have a map $g \in [E, P]$ such that for each $i, j \in I$,



Then the map g demonstrates that $(h, z) \in RS(E)$, since $(d_i g, z) = (g_i, z) \in G_i(E)$, and $h = f_i g_i = f_i d_i g = pg$.

($\supseteq$) If $(h, z) \in RS(E)$, then we have $E \xrightarrow{g} P$ such that $(d_i g, z) \in G_i(E)$, and $h = pg$. Then define, for each $i \in I$, $g_i = d_i g$, so $(g_i, z) \in G_i(E)$. Moreover, $h = pg = f_i d_i g = f_i g_i$, as required for $(h, z) \in LS(E)$. ∎

Proposition 4.4.5 *Suppose L is a CCD lattice in $\mathbf{set}^{\mathbf{C}^{op}}$. Then L satisfies wide Frobenius reciprocity.*

Proof Consider

$$
\begin{aligned}
& \sum_p \left(\bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\} \right) \\
= \; & \bigvee_C \sum_{DL(p)} \downarrow_P \left(\bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\} \right) \quad \text{(Prop. 4.2.5)} \\
= \; & \bigvee_C \sum_{DL(p)} \left(\bigcap \{\downarrow_P L(d_i)(y_i) : i \in I\} \right) \\
= \; & \bigvee_C \sum_{DL(p)} \left(\bigcap \{\mathcal{D}L(d_i) \downarrow_{D_i} (y_i) : i \in I\} \right) \quad \text{(naturality)} \\
= \; & \bigvee_C \left(\bigcap \{\Sigma_{DL(f_i)} \downarrow_{D_i} (y_i) : i \in I\} \right) \quad \text{(Lemma 4.4.4)} \\
= \; & \bigwedge_{L(C)} \left(\bigvee_C \{\Sigma_{DL(f_i)} \downarrow_{D_i} (y_i) : i \in I\} \right) \quad \text{(L is CCD)}
\end{aligned}
$$

$$= \bigwedge_{L(C)} \{\Sigma_{f_i}(y_i) : i \in I\} \text{ (Prop. 4.2.5)}$$

as required. ■

Unfortunately, to be able to use wide Frobenius reciprocity to characterize CCD lattices, we will need the category C to have wide pullbacks. While this is analagous to the characterization of sup lattices (which required C to have finite pullbacks), it is a far worse restriction. For $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ to be a topos, C must be small, so if C also has a terminal object, then the existence of wide pullbacks will mean C is both small and complete, and hence a poset (Mac Lane [5, p. 114]). Obviously, $\mathbf{C}$ being a poset is much more restrictive than simply asking that $\mathbf{C}$ have finite pullbacks. In the next chapter, we will attempt to remove the assumption of pullbacks, finite or otherwise, from these proofs.

Before we prove our characterization of CCD lattices, we need a lemma.

Lemma 4.4.6 *For any collection* $(x_i)_{i\in I} \subseteq L(C)$*,*

$$\bigvee_C \left[\bigcup \{\downarrow_C (x_i) : i \in I\}\right] = \bigvee_{L(C)} \{x_i : i \in I\}$$

Similarly,

$$\bigwedge_C \left[\bigcap \{\uparrow_C (x_i) : i \in I\}\right] = \bigwedge_{L(C)} \{x_i : i \in I\}$$

Proof We have

$$\begin{aligned}
& \bigvee_C \left[\bigcup \{\downarrow_C (x_i) : i \in I\}\right] \\
= & \bigvee_{L(C)} \{\Sigma_f(y) : \exists i \in i, (f, y) \in \downarrow_C (x_i)(D), D \in \mathbf{C}\} \\
= & \bigvee_{L(C)} \{\Sigma_f(y) : \exists i \in i, y \leq L(f)(x_i), D \in \mathbf{C}\} \\
= & \bigvee_{L(C)} \{\Sigma_f(y) : \exists i \in i, \Sigma_f(y) \leq x_i, D \in \mathbf{C}\} \\
= & \bigvee_{L(C)} \{x_i : i \in I\}
\end{aligned}$$

The other statement follows similarly. ■

Theorem 4.4.7 *If C has wide pullbacks, and L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, then L is also a CCD lattice if and only if*

1. *Each set $L(C)$ is a CCD lattice.*

2. *L satisfies wide Frobenius reciprocity.*

Proof ($\Rightarrow$) by Propositions 4.4.3 and 4.4.5.

($\Leftarrow$) We will prove this direction by showing the CCD equation

$$\bigvee_C \left(\bigcap_{i \in I} F_i \right) = \bigwedge_C \left\{ \bigvee_C F_i : i \in I \right\}$$

However, we need to interpret this equation a bit. The expression on the right is asking that we take the inf of a "set" of items. While this makes sense in **set**, it does not quite make sense in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, since the domain of $\bigwedge_C$ is $\mathcal{U}L(C)$. What is implicit in the right side of the expression above is that we first take the up-closure of the set $\{\bigvee_C F_i : i \in I\}$, then apply $\bigwedge_C$. One takes up-closure by up-closing each of the elements, then taking the intersection of the result. So the equation above is actually

$$\bigvee_C \left(\bigcap_{i \in I} F_i \right) = \bigwedge_C \left[\bigcap \left\{ \uparrow_C \bigvee_C F_i : i \in I \right\} \right]$$

which by the lemma we have just proven, reduces to

$$\bigvee_C \left(\bigcap_{i \in I} F_i \right) = \bigwedge_{L(C)} \left\{ \bigvee_C F_i : i \in I \right\}$$

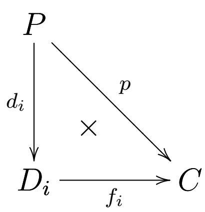
We shall demonstrate this last equation by showing containment in both directions.

($\subseteq$) $\bigcap_{i \in I} F_i \subseteq F_j$ for any $j \in I$, so $\bigvee_C \left(\bigcap_{i \in I} F_i \right) \subseteq \bigvee_C F_j$, hence $\bigvee_C \left(\bigcap_{i \in I} F_i \right) \subseteq \bigwedge_{L(C)} \{\bigvee_C F_j : j \in I\}$.

($\supseteq$) This direction is where we need to use the assumptions. For $F \in \mathcal{D}L(C)$, let $\zeta(F) = \{\Sigma_f y : (f, y) \in F(D), D \in \mathbf{C}\}$. Then consider

$$
\begin{aligned}
& \bigwedge_{L(C)} \left(\bigvee_{C} F_i : i \in I \right) \\
= \; & \bigwedge_{L(C)} \left(\bigvee_{L(C)} \zeta(F_i) : i \in I \right) \\
= \; & \bigwedge_{L(C)} \left(\bigvee_{L(C)} [\zeta(F_i)]^{\downarrow} : i \in I \right) \\
= \; & \bigvee_{L(C)} \left(\bigcap_{L(C)} [\zeta(F_i)]^{\downarrow} : i \in I \right) \text{(since L(C) is CCD)} \\
= \; & \bigvee_{L(C)} \left(\bigwedge_{L(C)} z_i : (z_i)_{i \in I} \in \Pi\zeta(F_i)^{\downarrow} \right) \text{(by lemma 2.3.2)} \\
= \; & \bigvee_{L(C)} \left(\bigwedge_{L(C)} \Sigma_{f_i}(y_i) : (\Sigma_{f_i}(y_i))_{i \in I} \in \Pi\zeta(F_i) \right) \text{(since we have a sup)} \\
= \; & \bigvee_{L(C)} \left(\sum_{p} \left(\bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\} \right) : (\Sigma_{f_i}(y_i))_{i \in I} \in \Pi\zeta(F_i) \right)
\end{aligned}
$$

where the last line follows by wide Frobenius, after taking the wide pullback of the f_i's:



So, it will be enough to show that for each collection $(\Sigma_{f_i}(y_i) \in F_i(D_i))_{i \in I}$, we have

$$
\sum_{p} \left(\bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\} \right) \leq \bigvee_{C} \left(\bigcap_{i \in I} F_i \right)
$$

Now for each i,

$$(f_i, y_i) \in F_i(D_i)$$

$$\begin{aligned}
&\Rightarrow \quad F(d_i)(f_i, y_i) \in F_i(P) \\
&\Rightarrow \quad (f_i d_i, L(d_i)(y_i)) \in F_i(P) \\
&\Rightarrow \quad (p, L(d_i)(y_i)) \in F_i(P) \\
&\Rightarrow \quad \left(p, \bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\}\right) \in F_i(P) \text{ since } F_i \text{ is down-closed}
\end{aligned}$$

So since this works for any i,

$$\left(p, \bigwedge_{L(P)} \{L(d_i)(y_i) : i \in I\}\right) \in \bigcap_{i \in I} F_i(P)$$

and hence $\sum_p \left(\bigwedge_{L(P)}\{L(d_i)(y_i) : i \in I\}\right)$ appears in the supremum $\bigvee_C \left(\bigcap_{i\in I} F_i\right)$. Thus $\sum_p \left(\bigwedge_{L(P)}\{L(d_i)(y_i) : i \in I\}\right) \leq \bigvee_C \left(\bigcap_{i\in I} F_i\right)$, as required. ■

Chapter 5

A 2-Categorical Approach

In this chapter, we will approach the material we have been looking at in a slightly different fashion. In Chapter 4, the proofs given were based on looking at what happened to the objects and arrows in **set**. In this chapter, we will look at diagrams in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, leading to a more 2-categorical approach to this area. The goal is to understand how we can remove the assumption that C has pullbacks from the proofs in the previous chapter.

Since we will be working in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, we will need to make a few notes about notation. Instead of writing

$$\mathbf{C}(-, D) \xrightarrow{\mathbf{C}(-,f)} \mathbf{C}(-, C) \quad \text{in } \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$$

for the embedding of an arrow in $\mathbf{C}$, $D \xrightarrow{f} C$, we will simply write

$$D \xrightarrow{\quad f \quad} C \qquad \text{in } \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$$

Moreover, if we have an element such as $x \in L(C)$, for $L \in \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, we will write the element as an arrow:

$$C \xrightarrow{\quad x \quad} L \qquad \text{in } \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$$

which we can do by Yoneda. Finally, note that if we have an $f \in [D, C]$, with $x \in L(C)$, then their composite $x(f)$ in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ is nothing more than $L(f)(x)$, ie.

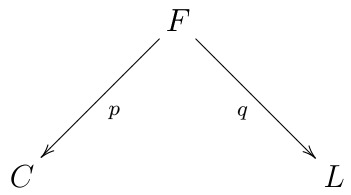
$$\begin{array}{ccc} D & \xrightarrow{f} & C \\ & \searrow^{L(f)(x)} & \downarrow x \\ & & L \end{array} \qquad \text{in } \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$$

5.1 Elements of the Down Object as Monic Spans

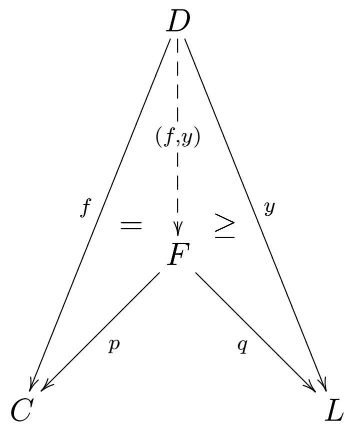
We will first look at how we can represent elements of the down object as a diagram in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Fix $C \in \mathbf{C}$, and order it (as an element of $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$) with the equality order; that is, for $f_1, f_2 \in [D, C]$, $f_1 \leq f_2 \Leftrightarrow f_1 = f_2$. Then we have:

$$\dfrac{\dfrac{F \in \mathcal{D}L(C)}{C \xrightarrow{F} \mathcal{D}L}}{C \overset{F}{\nrightarrow} L \text{ an order ideal}}$$

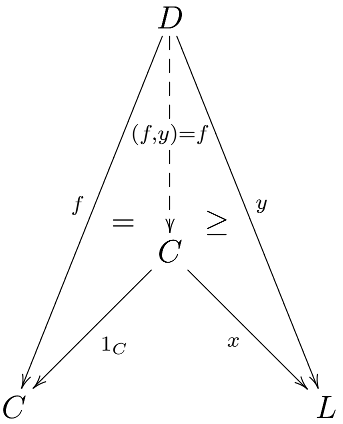
with the last equivalence holding by the universal property of the down object (Proposition 2.1.5). Thus an $F \in \mathcal{D}L(C)$ is a subobject of $L \times C$, up-closed in C, and down-closed in L. But since we have ordered C by equality, the condition "up-closed in C" is always satisfied. Moreover, the condition "down-closed in L" means that if we have $(f, y) \in F(D)$, and $y' \leq y$, then $(f, y') \in F(D)$. This is exactly the condition that F be a down-closed subfunctor that we saw in section 3.3. Thus this version of down-closed subfunctors is the same as we worked with previously, just expressed in a different way. We will write



for an $F \in \mathcal{D}L(C)$. Then $(f, y) \in F(D)$ means that the arrow (f, y) exists in the following diagram:



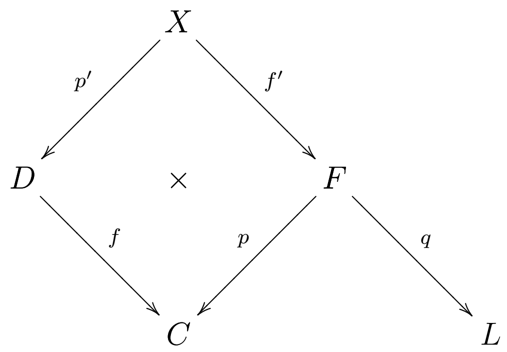
Example 5.1.1 *If we have an $x \in L(C)$, we could consider the monic span generated by 1_c and x. This gives:*



In other words, (f, y) is an element of this functor evaluated at D if and only if $y \leq L(f)(x)$. That is, the functor determined by 1_C and x is exactly the functor $\downarrow_C(x)$ that we saw in Section 3.3.

As a final note, we will consider the composition of a monic span with a representable arrow. This is done by pullback:

Proposition 5.1.2 *Suppose $F \xrightarrow{(p,q)} C \times L$ is a monic span, and $D \xrightarrow{f} C$ an arrow in $\mathbf{C}$. Let (X, p', f') be the pullback of f and p.*



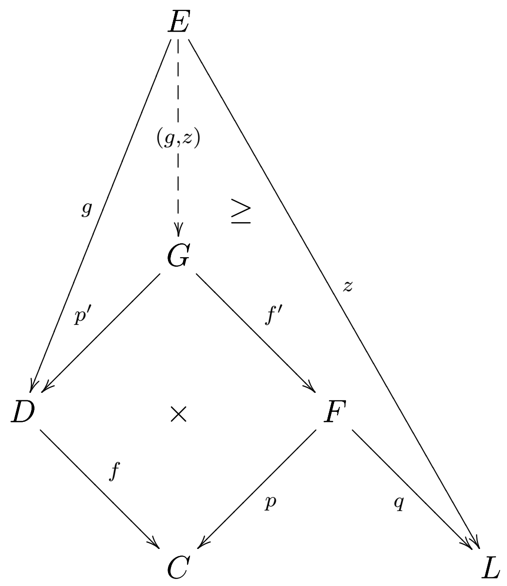
Then $X \xrightarrow{(p',\ qf')} D \times L$ is also a monic span. In this situation, we will call X the composite of F with f, and denote it by $F(f)$.

Proof Suppose we have two parallel arrows $Y \underset{b}{\overset{a}{\rightrightarrows}} X$, with $qf'a = qf'b$ and $p'a = p'b$. The last equation implies $fp'a = f'pb$, or $pf'a = pf'b$. Since (p, q) is jointly monic, this implies $f'a = f'b$. Then $f'a = f'b$, together with $p'a = p'b$, implies $a = b$, by the pullback property. Thus indeed (p', qf') is jointly monic. ■

In fact, if F is in $\mathcal{D}L(C)$, then $F(f)$ is in $\mathcal{D}L(D)$. To demonstrate this, we shall show that if $F \in \mathcal{D}L(C)$, $F(f)$ is actually something we have seen before: the functor $\mathcal{D}L(f)(F)$.

Proposition 5.1.3 *Let $f \in [D, C]$, and $F \in \mathcal{D}L(C)$, say $F \xrightarrow{(p,q)} C \times L$. Then the composite $F(f)$ is equal to the functor $\mathcal{D}L(f)(F)$.*

Proof Let $G \xrightarrow{(p',\, qf')} D \times L$ be the composite $F(f)$.



We need to show that $(g, z) \in G(E) \iff (fg, z) \in F(E)$, since that is the action of $\mathcal{D}L(f)$.

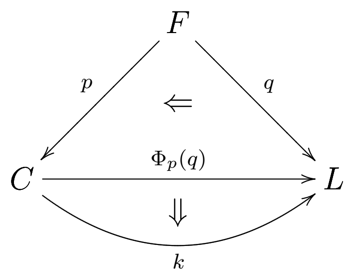
($\Rightarrow$) If $(g, z) \in G(E)$, then we have an arrow (g, z) as above. We claim that the composition $f'(g, z)$ witnesses that $(fg, z) \in F(E)$. Indeed, $pf'(g, z) = fp'(g, z) = fg$, and $qf'(g, z) \geq z$ since $(g, z) \in G(E)$.

($\Leftarrow$) If $(g, z) \in G(E)$, then we have an arrow $E \xrightarrow{(fg,z)} F$ which witnesses it. By the pullback, we get an arrow $E \xrightarrow{i} G$. We claim it witnesses $(g, z) \in G(E)$. Indeed $p'i = g$, and $qf'i = q(fg, z) \geq z$, both by the property of the arrow i. ■

5.2 $\bigvee$ as a Kan Extension

Writing a subfunctor as a monic span allows us to view the sup map in $\mathbf{set}^{\mathbf{C}^{\text{op}}}$ in a different way. Specifically, consider the subfunctor/monic span $F \xrightarrow{(p,q)} C$. If L is a sup lattice, then L is complete, and since F and C are posets, the left Kan extension of q along p (which we shall write as $\Phi_p(q)$) exists.

Recall that being a left Kan extension for q along p here means that $q \leq [\Phi_p(q)]p$, and $\Phi_p(q)$ is universal with that property; that is, whenever we have a k such that $q \leq kp$, then $\Phi_p(q) \leq k$:



The key point of this chapter is the following fact relating the Kan extension to the sup map:

Proposition 5.2.1 *If L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$, with F and $\Phi_p(q)$ as above, then*

$$\bigvee_C F = \Phi_p(q)$$

Proof To demonstrate that $\Phi_p(q) = \bigvee_C(F)$, we shall use the following fact: for two parallel arrows $X \underset{x_1}{\overset{x_1}{\rightrightarrows}} L$ in $\mathbf{set}^{\mathbf{C}^{\text{op}}}$,

$$(x_1 = x_2) \Longleftrightarrow \forall D \in \mathbf{C}, \forall X \xrightarrow{f} C, x_1 f = x_2 f.$$

Fix an arrow $f \in [D, C]$. By the co-end formula for Kan extensions,

$$\begin{aligned}
[\Phi_p(q)](f) &= \bigvee \{q(f', y) : (f', y) \in F(D), p(f', y) \leq f\} \\
&= \bigvee \{q(f', y) : (f', y) \in F(D), f' \leq f\} \\
&= \bigvee \{y : (f, y) \in F(D)\} \text{ (since } \leq \text{ on } \mathbf{C} \text{ is equality)}
\end{aligned}$$

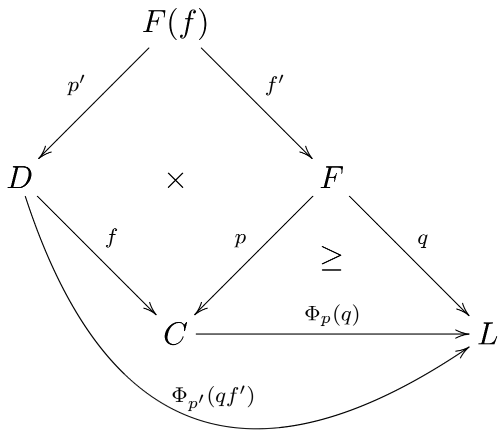
On the other hand,

$$\begin{aligned}\left(\bigvee_C F\right) f &= \left(\bigvee_D \mathcal{D}L(f)F\right) \text{ (by naturality)} \\ &= \bigvee\{\Sigma_g(z) : (g,z) \in \mathcal{D}L(f)(F)(E), E \in \mathbf{C}\} \\ &= \bigvee\{\Sigma_g(z) : (fg,z) \in F(E), E \in \mathbf{C}\}\end{aligned}$$

Taking $g = 1_D$, $z = y$, we can see that every element of the sup for $\Phi_p(q)$ is in the above sup, so $\Phi_p(q)(f) \leq (\bigvee_C F)\, f$.

Conversely, for any $(fg, z) \in F(E)$, $z \leq [\Phi_p(q)](fg)$, so $z \leq L(g)[\Phi_p(q)](f)$, and hence $\Sigma_g(z) \leq [\Phi_p(q)](f)$, demonstrating that $(\bigvee_C F)\, f \leq [\Phi_p(q)](f)$. ■

We can now express the naturality of $\bigvee$ in a different format. The naturality equation is $L(f) \bigvee_C F = \bigvee_D \mathcal{D}L(f)(F)$. We now know that $\bigvee_C(F) = \Phi_p(q)$, and as we saw earlier, $\mathcal{D}L(f)(F)$ is just $F(f)$ in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. So the naturality equation can be expressed as the equality of $[\Phi_p(q)]f$ and $\Phi_{p'}(qf')$:



Since this has the look of a Beck-Chevalley condition, we shall use the following definition:

Definition Suppose that L is a poset in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, and each $F \xrightarrow{(p,q)} C \times L$ in $\mathcal{D}L(C)$ has a Kan extension $\Phi_p(q)$. Say that L satisfies *global BC* if for all $C, D \in \mathbf{C}$, $F \in \mathcal{D}L(C)$, and $f \in [D, C]$, $[\Phi_p(q)]f = \Phi_{p'}(qf')$.

The reason for calling this Beck-Chevalley condition "global" will be seen in the next section. We have now compiled enough information to prove the following theorem:

Theorem 5.2.2 *For L a poset in* $\mathbf{set}^{\mathbf{C}^{op}}$, *$L$ is a sup lattice if and only if*

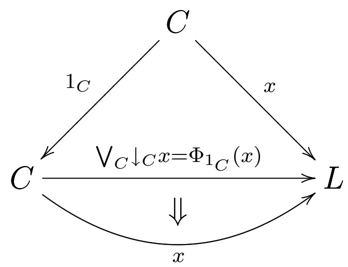
1. $\forall C \in \mathbf{C}, \forall F \xrightarrow{(p,q)} C \times L \in \mathcal{D}L(C)$, *the Kan extension* $\Phi_p(q)$ *exists.*

2. *L satisfies global BC.*

Proof ($\Rightarrow$) The Kan extensions exist, as noted in the remarks before Proposition 5.2.1, by the completeness of L. Moreover, the naturality of $\bigvee$ implies that each $\Phi_p(q)$ satisfies global BC, as seen above.

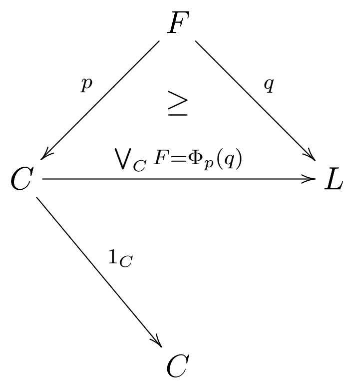
($\Leftarrow$) Given this information, define $\bigvee_C F := \Phi_p(q)$. Then $\bigvee$ is natural by global BC.

To check that it is left adjoint to $\downarrow$, we need to show the co-unit $\forall C \in \mathbf{C}, \forall x \in L(C), \bigvee_C \downarrow_C x \leq x$, and the unit $\forall F \in \mathcal{D}L(C), F \subseteq \downarrow_C \bigvee_C F$. For the co-unit, if we recall the characterization of $\downarrow_C (x)$ seen in Example 5.1.1, then we have:



where the $\Downarrow$ indicates that $\bigvee_C \downarrow_C x \leq x$ by the universal property of $\Phi_{1_C}(x)$.

For the unit, consider

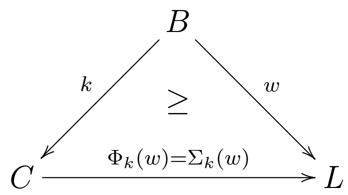


which demonstrates that $F \subseteq \downarrow_C \bigvee_C F$. ∎

5.3 Σ as a Kan Extension

Recall that in Section 4.2, we defined $\bigvee_C F$ to be the supremum of $\Sigma_k(w)$, where $(k, w) \in F(B)$, $B \in \mathbf{C}$. Since we have just shown that the map $\bigvee_C F$ is a Kan extension, it makes sense that the "components" of $\bigvee_C F$, the $\Sigma_k(w)$'s, should also be Kan extensions.

Proposition 5.3.1 *Let L be a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, and $B \xrightarrow{k} C$ in $\mathbf{C}$. If $L(k)$ has a left adjoint Σ_k, then for each $w \in L(B)$, the Kan extension of w along k, $\Phi_k(w)$, exists and equals $\Sigma_k(w)$:*

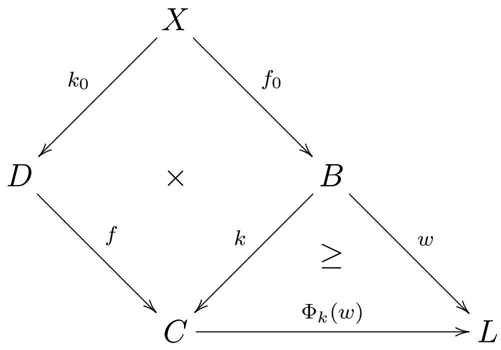


Proof We first need to show that $w \leq [\Sigma_k(w)]k$. But since $[\Sigma_k(w)]k = L(k)[\Sigma_k(w)]$, this is nothing more than the unit of the adjunction $\Sigma_k \dashv L(k)$.

We also need to show it is universal with this property. Suppose we have an $x \in L(C)$ such that $w \leq xk$. Again, $xk = L(k)x$, so by the co-unit of the adjunction, $w \leq xk$ becomes $\Sigma_k(w) \leq x$. Hence $\Sigma_k(w)$ is universal among x's with the property that $w \leq xk$. ■

Recall that in the previous section we found a "global BC" condition for the Kan extensions $\bigvee_C F$. Now that we have expressed the Σ's as Kan extensions, we can similarly define a "local BC" condition for these "local" Kan extensions.

Definition Suppose that L is a poset in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, and $\forall k \in [B, C]$, $w \in L(C)$, the Kan extension $\Phi_k(w)$ exists. Then say that L satisfies *local BC* if, for each (k, w) with the pullback diagram



we have $\Phi_{k_0}(wf_0)$ exists and equals $[\Phi_k(w)]f$.

Notice the difference between global BC and local BC. In global BC, we knew that the Kan extension $\Phi_{p'}(wf')$ existed. Here, however, we do not know a priori that the Kan extension $\Phi_{k_0}(wf_0)$ exists, since X is not necessarily representable.

However, if C has pullbacks, then X is representable, and we have the following proposition:

Proposition 5.3.2 *Suppose C has finite pullbacks, L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, and each $L(k)$ has a left adjoint Σ_k. Then L satisfies local BC if and only if each adjunction $\Sigma_k \dashv L(k)$ satisfies Beck-Chevalley.*

Proof As above, let (X, k_0, f_0) be the pullback of f and k. Then the adjunction $\Sigma_k \dashv L(k)$ satisfies Beck-Chevalley if and only if for each $w \in L(B)$, $\Sigma_{k_0}(L(f_0)w) = L(f)\Sigma_k(w)$. But X is representable, so by proposition 5.3.1,

$$\frac{\Sigma_{k_0}(L(f_0)w) = L(f)\Sigma_k(w)}{\Phi_{k_0}(wf_0) = [\Phi_k(w)]f}$$

which is local BC. ∎

Thus local BC is a generalization of the condition "each adjunction $\Sigma_k \dashv L(k)$ satisfies Beck-Chevalley". However, the difference is that we can use local BC *even when the category C does not have pullbacks.* Thus our next result will be to show that we can replace the Beck-Chevalley condition with local BC to give a characterization of sup lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ that does not require that the category C have pullbacks.

5.4 Sup Lattices in $\mathbf{set}^{\mathbf{C}^{\mathbf{op}}}$ Revisited

Our task is to prove the following theorem:

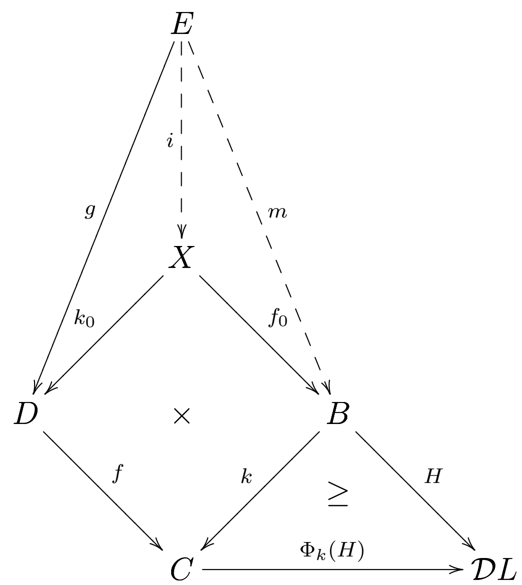
Theorem 5.4.1 *For C any small category, L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{op}}$ if and only if*

1. *For each $C \in \mathbf{C}$, $L(C)$ is a sup lattice.*
2. *For each $D \xrightarrow{f} C$ in $\mathbf{C}$, $L(f)$ has left and right adjoints Σ_f and Π_f.*
3. *L satisfies local BC.*

As we did in Chapter 4, we shall demonstrate (⇒) by going through the lattice $\mathcal{D}L$. Thus our first result is

Lemma 5.4.2 *If L is a poset in $\mathbf{set}^{\mathbf{C}^{op}}$, then $\mathcal{D}L$ satisfies local BC.*

Proof Suppose $H \in \mathcal{D}L(B)$, with morphisms $B \xrightarrow{k} C$ and $D \xrightarrow{f} C$ in $\mathbf{C}$. We then need $(\Phi_k H)f$ to be the left Kan extension of Hf_0 along k_0.



First, note that

$$[[\Phi_k H]f]k_0 = [\Phi_k H]kf_0 \geq Hf_0$$

so $[\Phi_k H]f$ has the required property.

We now need it to be universal, so assume we have $J \in \mathcal{D}L(D)$ such that $Jk_0 \supseteq Hf_0$. We need to show $[\Phi_k H]f \leq J$. That is, $(g,z) \in [(\Phi_k H)f](E)$ must imply $(g,z) \in J(E)$. Indeed:

$$\frac{\dfrac{(g,z) \in [(\Phi_k H)f](E)}{(fg,z) \in [\Phi_k H](E)}}{\exists m \in [E,B] \text{ s.t. } (m,z) \in H(E), \text{ and } fg = km}$$

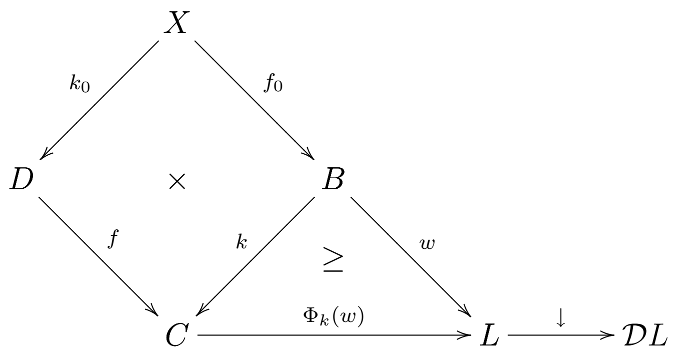
With the last equivalence by the definition of $\Sigma_{\mathcal{D}L(k)}(H)$ (which is equal to $\Phi_k H$). Then by the pullback, $\exists i \in [E,X]$ such that $m = f_0 i$ and $g = k_0 i$. Then $(i,z) \in Hf_0(E)$ since $(f_0 i, z) = (m,z) \in H(E)$.

But $Hf_0 \subseteq Jk_0$, so $(i,z) \in Jk_0(E)$. Hence $(g,z) = (k_0 i, z) \in J(E)$. Thus $[\Phi_k H]f \leq J$, as required. ■

As a result, we can demonstrate that each sup lattice L has local BC.

Proposition 5.4.3 *Suppose L is a sup lattice. Then L satisfies local BC.*

Proof Given



we need to show $[\Phi_k(w)]f$ is $\Phi_{k_0}(wf_0)$. Note that

$$[[\Phi_k w]f]k_0 = [\Phi_k w]kf_0 \geq wf_0$$

so $[\Phi_k(w)]f$ has the required property.

We also need it to be universal. Suppose we have $y \in L(D)$ such that $yk_0 \geq wf_0$. Then

$$\begin{aligned} & yk_0 \geq wf_0 \\ \Rightarrow\ & (\downarrow y)k_0 \geq (\downarrow w)f_0 \\ \Rightarrow\ & \downarrow y \geq \Phi_{k_0}(\downarrow wf_0) \text{ (universal property)} \\ \Rightarrow\ & \bigvee \downarrow y \geq \bigvee \Phi_{k_0}(\downarrow wf_0) \\ \Rightarrow\ & y \geq \bigvee \Phi_k(\downarrow wf) \text{ (by lemma 5.4.2)} \\ \Rightarrow\ & y \geq [\Phi_k(w)]f \text{ (Proposition 4.2.5)} \end{aligned}$$

So $[\Phi_k(w)]f$ has the required universal property. ■

We can now prove Theorem 5.4.1. The difficult direction will be ($\Leftarrow$).

Proof ($\Rightarrow$) by Propositions 4.2.2, 4.2.5, 4.2.6, and 5.4.3.

($\Leftarrow$) We will use Theorem 5.2.2 to show this direction. By that theorem, L will be a sup lattice if we can show that the global Kan extensions $\Phi_p(q)$ exist and satisfy global BC.

Existence Fix an $F \xrightarrow{(p,q)} C \times L$ in $\mathcal{D}L(C)$. We need to show $\Phi_p(q)$ exists. We will in fact show that $\bigvee\{\Phi_h(z) : (h,z) \in F(E), E \in \mathbf{C}\}$ is the required Kan extension.

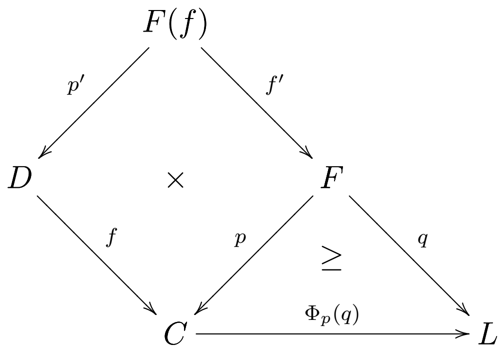
We first need to show that $q \leq [\bigvee\{\Phi_h(z) : (h,z) \in F(E), E \in \mathbf{C}\}]\, p$. Let $(h,z) \in F(E)$. Then

$$\begin{aligned} q(h,z) &= z \\ &\leq [\Phi_h(z)]h \\ &= [\Phi_h(z)]p(h,z) \\ &\leq \left[\bigvee\{\Phi_h(z) : (h,z) \in F(E), E \in \mathbf{C}\}\right] p(h,z) \end{aligned}$$

Thus $q \leq [\bigvee\{\Phi_h(z) : (h,z) \in F(E), E \in \mathbf{C}\}]\, p$.

For universality, suppose there exists an $x \in L(C)$ with the property that $q \leq xp$. Then for each $(h,z) \in F(E)$, $q(h,z) \leq xp(h,z)$, ie., $z \leq kh$. Thus $\Phi_h(z) \leq x$ for all $(h,z) \in F(E)$. Thus $\bigvee\{\Phi_h(z) : (h,z) \in F(E), E \in \mathbf{C}\} \leq x$, so the universal property holds.

Global BC Given:



we need to show

$$[\Phi_p(q)]f = \Phi_{p'}(qf')$$

We will do this by showing inequality in both directions.

($\geq$) Consider:

$$[\Phi_p(q)]fp' = [\Phi_p(q)]pf' \geq qf'$$

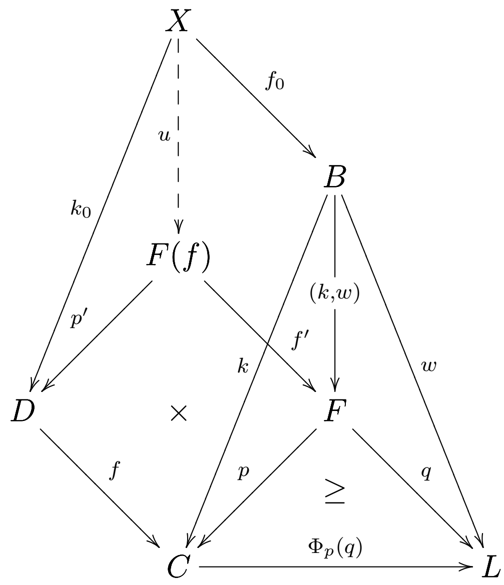
Thus $[\Phi_p(q)]f$ has the same property as $\Phi_{p'}(qf')$. Then the universality of $\Phi_{p'}(qf')$ implies that $[\Phi_p(q)]f \geq \Phi_{p'}(qf')$.

($\leq$) Conversely, consider

$$\begin{aligned}
& [\Phi_p(q)]f \\
= & \left[\bigvee\{\Phi_k(w) : (k, w) \in F(B), B \in \mathbf{C}\}\right] f \\
= & \bigvee\{[\Phi_k(w)]f : (k, w) \in F(B), B \in \mathbf{C}\} \text{ (L(f) has a right adjoint)} \\
= & \bigvee\{[\Phi_{k_0}(wf_0)] : (k, w) \in F(B), B \in \mathbf{C}\} \text{ (by local BC)}
\end{aligned}$$

So we only need to show that for each $(k, w) \in F(B)$, $\Phi_{k_0}(wf_0) \leq \Phi_{p'}(qf')$. To do this, we will show that $\Phi_{p'}(qf')$ has the property of $\Phi_{k_0}(wf_0)$, that is, $wf_0 \leq [\Phi_{p'}(qf')]k_0$.

Let (X, f_0, k_0) be the pullback of (f, k). Then we have the following diagram:



Since $p(k, w)f_0 = kf_0 = fk_0$, the pullback gives the existence of u. Then consider

$$\begin{aligned} & [\Phi_{p'}(qf')]k_0 \\ = \; & [\Phi_{p'}(qf')]p'u \\ \geq \; & qf'u \\ = \; & qf_0(k, w) \\ = \; & wf_0 \end{aligned}$$

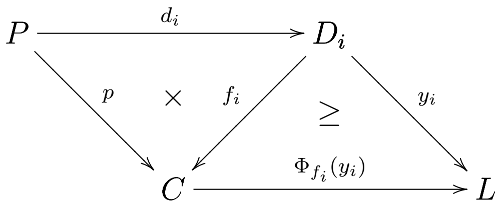
as required. Thus the universal property of $\Phi_{k_0}(wf_0)$ gives $\Phi_{k_0}(wf_0) \leq \Phi_{p'}(qf')$, and the result follows. ■

Thus we have characterized sup lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ without assuming that the category C had pullbacks.

5.5 CCD Lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ Revisited

In Section 5.3, we found a generalization of the Beck-Chevalley condition that could be used without assuming that **C** had pullbacks. Similarly, we can define a generalization of wide Frobenius reciprocity that can be used without assuming **C** has wide pullbacks.

Definition Suppose that L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Then say that L satisfies *local wide FR* if for any set of maps $(D_i \xrightarrow{f_i} C)_{i\in I}$ in $\mathbf{C}$, and any collection $(y_i \in L(D_i))_{i\in I}$, with $(P, (d_i)_{i\in I}) \in \mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ the wide pullback of the f_i's:



the Kan extension $\Phi_p\left(\bigwedge\{y_i d_i : i \in I\}\right)$ exists and equals $\bigwedge\{\Phi_{f_i}(y_i) : i \in I\}$.

Analagously to the previous situation, if $\mathbf{C}$ has wide pullbacks, then local wide FR is equivalent to wide Frobenius reciprocity.

Proposition 5.5.1 *Suppose that* $\mathbf{C}$ *has wide pullbacks, and* L *is a sup lattice in* $\mathbf{set}^{\mathbf{C}^{op}}$*. Then* L *satisfies local wide FR if and only if* L *has wide Frobenius reciprocity.*

Proof Since $\mathbf{C}$ has wide pullbacks, the P in the above diagram is representable. Then the equality $\Sigma_{f_i}(y_i) = \Phi_{f_i}(y_i)$ proves the equivalence of the two definitions. ■

If $\mathbf{C}$ does not have wide pullbacks, we can still use local wide FR, since the wide pullback of representables always exists in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Thus our conjecture for CCD lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ is:

Conjecture 5.5.2 *For* $\mathbf{C}$ *any small category, and* L *a sup lattice in* $\mathbf{set}^{\mathbf{C}^{op}}$*,* L *is also a CCD lattice if and only if*

1. *Each set* $L(C)$ *is a CCD lattice.*

2. L *satisfies local wide FR.*

Chapter 6

Conclusion

The major unresolved question in this work is the proof of the final theorem. However, the groundwork is present to complete the proof. We already know that L being CCD implies each $L(C)$ is CCD, and that L satisfies wide Frobenius reciprocity. A minor modification should allow one to prove that L being CCD implies L has local wide FR. For the other direction, one simply has to follow the method of proof for the characterization of sup lattices: find a definition of "global wide FR", and show its equivalence to the CCD equation.

With that completed, one could next wonder where else wide Frobenius reciprocity could be used. It may be useful in other areas in which Frobenius reciprocity occurs. Perhaps other results which involve Frobenius reciprocity could be given infinite versions by applying wide Frobenius reciprocity?

A second interesting notion is the idea of looking at $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ for C a poset. We have shown that the characterization of CCD lattices is simpler and easier to understand if C is a poset. Are there other areas where looking at $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ for $\mathbf{C}$ a poset makes results clearer?

Finally, one could attempt to apply one of the main methods of this thesis to other toposes. Specifically, the method of characterizing lattice structures L by looking at L's object of subobjects $\mathcal{D}L$. $\mathcal{D}L$ has two attractive properties. As we know, it is always a CCD lattice; thus any properties we would ask of L, $\mathcal{D}L$ must satisfy. Secondly, as has been demonstrated by many results in this thesis, $\mathcal{D}L$ can often be easier to work with than the original L. The method used in this thesis is thus both powerful and practical.

Of course, one could not use this method to look at CD lattices, since $\mathcal{D}L$ or $\mathcal{P}L$ is not CD unless the topos has choice. Thus, the usefulness of this method demonstrates another advantage to working with CCD lattices instead of CD lattices in a general topos.

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