

A unified framework for generalized multicategories

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(joint work with Mike Shulman)

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Generalized multicategories in the literature

There have been many instances of work on generalized multicategories:

- Albert Burroni: T -catégories (1971);
- Claudio Hermida: Lax Bimod(T)-algebras (2001);
- Tom Leinster: T -categories (2004);
- Maria Manuel Clementino, Dirk Hofman, Walter Tholen: (T, V) -algebras (2003-),

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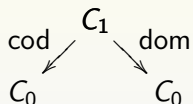
and others: each of these has many interesting examples.

Our goal today is to briefly give a framework in which all of these various notions of generalized multicategories are unified.

Main idea: use a type of double category rather than bicategories.

What are categories?

Let's start with categories. A category can be seen as a span of sets $C_0 \xrightarrow{\mathbf{C}} C_0$:



then we can view the composition operation as a span morphism $\mathbf{C}^2 \rightarrow \mathbf{C}$, and the identity as a span morphism $1_{C_0} \rightarrow \mathbf{C}$; the axioms for a category define how these morphisms interact.

This leads to the following definition:

Categories

Definition

A **monoid** in a bicategory \mathbf{B} consists of an object X , an arrow $X \xrightarrow{\mathbf{C}} X$, and 2-cells $\mathbf{C}^2 \rightarrow \mathbf{C}$, $1_X \rightarrow \mathbf{C}$ (plus axioms).

We have:

- a monoid in $\text{Span}(\text{Set})$ is a category;
- a monoid in the bicategory of \mathbf{V} -matrices is a \mathbf{V} -category;
- a monoid in the bicategory of spans for an arbitrary category \mathbf{C} with pullbacks is an internal \mathbf{C} -category.

Notice that the idea of a monoid in a bicategory covers not only categories, but also “generalized” categories (enriched and internal). But there is a problem:

Functors?

If a (generalized) category is a monoid in some bicategory, what is a functor?

Functors?

If a (generalized) category is a monoid in some bicategory, what is a functor?

No easy answer. A functor uses functions, not spans. We need access to to another type of arrow between the objects of the bicategory:

- for $\text{Span}(\text{set})$, we need functions;
- for $\mathbf{V}\text{-mat}$, we also need functions;
- for $\text{Span}(\mathbf{C})$, we need the arrows of \mathbf{C} .

So, instead of bicategories, we move to double categories. In fact, we will move one step further:

Virtual double categories

Definition

A **virtual double category** consists of:

- a set of objects and vertical arrows (forming a category),
- a set of horizontal arrows (no composites assumed),
- cells of the form:

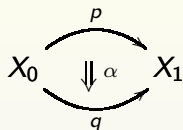
$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{p_1} & X_1 & \xrightarrow{p_2} & X_2 & \xrightarrow{p_3} & \dots & \xrightarrow{p_n} & X_n \\
 \downarrow f & & & & \Downarrow \alpha & & & & \downarrow g \\
 Y_0 & \xrightarrow{\quad\quad\quad} & & & & & & & Y_1 \\
 & & & & & & & & \downarrow q
 \end{array}$$

as well as “unit” cells with no horizontal domain.

Think of the vertical arrows as things like functions, and the horizontal arrows as things like spans.

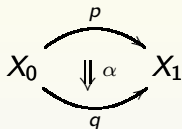
Bicategories vs. virtual double categories

So a bicategory has data like

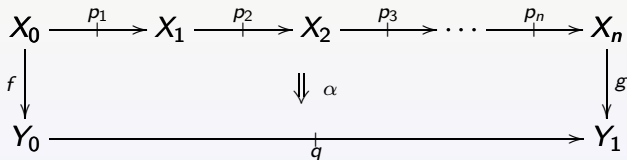


Bicategories vs. virtual double categories

So a bicategory has data like



whereas a virtual double category has data like



Monoids in a virtual double category

Definition

A **monoid** in a virtual double category consists of a horizontal cell

$X \xrightarrow{p} X$ and cells

$$\begin{array}{ccccc}
 X & \xrightarrow{p} & X & \xrightarrow{p} & X \\
 \parallel & & \Downarrow \bar{x} & & \parallel \\
 X & \xrightarrow{p} & X & & X
 \end{array}$$

and

$$\begin{array}{ccc}
 & X & \\
 & \Downarrow \hat{x} & \\
 X & \xrightarrow{p} & X
 \end{array}$$

satisfying associativity and identity axioms.

Monoid morphisms

Definition

A **monoid morphism** between monoids (X, p) and (Y, q) consists of a vertical arrow $X \xrightarrow{f} Y$ and a cell

$$\begin{array}{ccc}
 X & \xrightarrow{p} & X \\
 f \downarrow & \Downarrow \bar{f} & \downarrow f \\
 Y & \xrightarrow{q} & Y
 \end{array}$$

(we can also define monoid bimodules that correspond to profunctors).

The $\mathbb{M}od$ construction

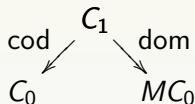
Call this construction $\mathbb{M}od$: it takes any virtual double category and produces a new virtual double category. $\mathbb{M}od$ takes:

- (Sets, functions, spans) \mapsto (Categories, functors, profunctors);
- (Sets, functions, V -matrices) \mapsto (V -categories, V -functors, V -profunctors);
- (objects of \mathbf{C} , arrows of \mathbf{C} , spans of \mathbf{C} \mapsto (internal \mathbf{C} -categories, internal \mathbf{C} -functors, internal \mathbf{C} -natural transformations).

Notice there are no other assumptions, i.e. we don't ask for composites of profunctors.

What are multicategories?

The arrows of a multicategory have as their domain *multiple* object. So, the data for a multicategory can be seen as a span $C_0 \longrightarrow MC_0$:



where M is the free monoid monad. The composition and identities have a similar expression as a morphism of spans, but we need to use the multiplication and unit of the monad M .

Generalized multicategories: idea

The realization with generalized multicategories: replace M by some other arbitrary “monad”, get other interesting things. For example:

- (Barr) the ultrafilter monad on sets: topological spaces;
- other examples include symmetric multicategories, braided multicategories, Lawvere theories, globular operads, graded categories, approach spaces, etc.

Monads in what sense?

Problems:

- in what sense are these monads?
- what does it mean “take monoids” when we modify the horizontal cells?

They appear to be monads on a bicategory. But there is a problem: some monads are actually lax. But there is no 2-category of bicategories, lax functors, and lax (or op-lax) transformations, so there is no sense in which these could be monads in some 2-category.

How have others gotten around this? Work with only *particular* bicategories (spans, matrices, or profunctors) and make particular definitions.

Virtual double categories to the rescue again!

But, when we work with the relevant virtual double categories rather than bicategories, we *can* define monads on them. In particular:

- there is a 2-category of virtual double categories, functors, and transformations (because the transformations have vertical components);
- thus, there is a well-defined notion of “monad on a virtual double category”;
- the definition of “monad” used in other papers is often very closely related to this notion.

Horizontal kleisli construction

Given a monad T on a virtual double category \mathbb{X} , define a new virtual double category $\mathbb{H}\text{-Kl}(\mathbb{X}, T)$ with

- objects those of \mathbb{X} ;
- vertical arrows those of \mathbb{X} ;
- a horizontal arrow $X \xrightarrow{P} Y$ is a horizontal arrow $X \xrightarrow{P} TY$ in \mathbb{X} ;
- cells are defined using the unit and multiplication of T .

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Even if horizontal composites exist in \mathbb{X} , they need not exist in $\mathbb{H}\text{-Kl}(\mathbb{X}, T)$! This the main reason for using *virtual* double categories.

Generalized multicategories: definition

Given a monad T on a virtual double category \mathbb{X} , the virtual double category of generalized T -multicategories is given by

- applying $\mathbb{H}\text{-Kl}(\mathbb{X}, T)$,
- then applying $\mathbb{M}\text{od}$.

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We get a new virtual double category with:

- objects generalized multicategories,
- vertical arrows multicategory functors,
- horizontal arrows multicategory profunctors.

Includes a wealth of examples.

Conclusion

Solves three problems at once:

- gives an easy definition of the functors (and profunctors) between generalized multicategories;
- shows how the input for a generalized multicategory is a monad;
- deals with lack of horizontal composition.

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- deals with lack of horizontal composition.

This construction also

- unifies previous definitions;
- splits the construction into two parts, making it more easily analysable.

Paper

Lots more on this subject:

- different types of generalized multicategories;
- when $\mathbb{H}\text{-KI}(\mathbb{X}, T)$ has composites;
- representable generalized multicategories;

as well as comparisons to previous theories, can be found in:

- “A unified framework for generalized multicategories”
(Cruttwell and Shulman), available at [arxiv.org:0907.2460](https://arxiv.org/abs/0907.2460).