Structures in tangent categories

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Outline

- What are tangent categories?
  - Definitions: intuitive and more precise.
  - Examples.
- What can one do within a tangent category?
  - Vector fields and their Lie bracket.
  - Vector spaces.
  - Vector bundles.
  - Differential forms.
  - Connections on a vector bundle.

Tangent categories (intuitively)

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

(Intuitively) A tangent category consists of a category $\mathbb{X}$, which has, for each object M, an associated bundle over M, called TM, with the following properties:

- each TM is an additive bundle over M, in a natural way;
- each TM is a "vector" bundle over M, in a natural way;
- T "preserves the structure of each bundle TM" in a natural way.

Tangent categories (more precisely)

Definition

More specifically, this means we have a functor $T : \mathbb{X} \longrightarrow \mathbb{X}$, with:

- (projection) a natural transformation $p : T \longrightarrow I$;
- (addition and zeroes) natural transformations $+ : T_2 \longrightarrow T$ and $0 : I \longrightarrow T$;
- (vertical lift) a natural transformation $\ell : T \longrightarrow T^2$ satisfying a certain universality property;
- (canonical flip) a natural transformation $c : T^2 \longrightarrow T^2$;
- a number of coherence axioms.

Tangent category examples

- *(i)* The canonical example: finite dimensional smooth manifolds.
- *(ii)* Convenient manifolds (with the kinematic tangent bundle).
- *(iii)* Any Cartesian differential category.
- *(iv)* The infinitesimally linear objects in a model of synthetic differential geometry.
- *(v)* Commutative ri(n)gs and its opposite (and other associated categories in algebraic geometry).
- *(vi)* The category of C-∞ rings.
- *(vii)* (Lack/Leung) A category of Weyl algebras.
- *(viii)* (Rosický) If $\mathbb{X}$ has tangent structure, then so does each slice $\mathbb{X}/M$.

Vector fields and their Lie bracket

Definition

If $(\mathbb{X}, T)$ is a tangent category with an object $M \in \mathbb{X}$, a **vector field on** M is a map $M \xrightarrow{v} TM$ with $pv = 1$.

If $\mathbb{X}$ has negation, given two vector fields $v_1, v_2 : M \longrightarrow TM$, Rosický showed how to use the universal property of vertical lift to define the Lie bracket vector field $[v_1, v_2] : M \longrightarrow TM$ so that the Jacobi identity

$$[v_1, [v_2, v_3] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0$$

is satisfied.

Vector Spaces/Differential objects

Vector spaces in tangent categories are represented by objects whose tangent bundle is trivial:

Definition

A **differential object** in a tangent category consists of a commutative monoid (A, σ, ζ) with a map $\hat{p} : TA \longrightarrow A$ such that

$$A \xleftarrow{\hat{p}} TA \xrightarrow{p} A$$

is a product diagram, so that $TA \cong A \times A$ (as well as some additional coherence axioms).

- $\mathbb{R}^n$'s in the category of smooth manifolds.
- The pullback of $p : TM \longrightarrow M$ along a point of M.
- If T is representable with representing object D, get an associated ring R which is differential (thus satisfying the "Kock-lawvere" axiom).

Vector/Differential bundles (intuition)

In general:

- a group bundle is a group in $\mathbb{X}/M$;
- a vector bundle is a vector space in $\mathbb{X}/M$;
- so a differential bundle should be a differential object in the canonical tangent category structure on $\mathbb{X}/M$.

Vector/Differential bundles (more precisely)

Definition

A **differential bundle** in a tangent category consists of an additive bundle $q : E \longrightarrow M$ with a map $\lambda : E \longrightarrow TE$ so that $q : E \longrightarrow M$ becomes a differential object in the slice tangent category $\mathbb{X}/M$.

(i) If A is a differential object, then for each object M, $\pi_2 : A \times M \longrightarrow M$ is a differential bundle.

(ii) For each object M, $p : TM \longrightarrow M$ is a differential bundle.

(iii) The pullback of a differential bundle $q : E \longrightarrow M$ along any map $f : X \longrightarrow M$ is a differential bundle.

(iv) If $q : E \longrightarrow M$ is a differential bundle, $T(q) : TE \longrightarrow TM$ is also.

Linear maps between differential bundles

- A **morphism of differential bundles** between differential bundles $(q : E \longrightarrow M)$, $(q' : E' \longrightarrow M')$ is simply a pair of maps $f : E \longrightarrow E'$, $g : M \longrightarrow M'$ making the obvious diagram commute.
- A morphism of differential bundles (f, g) is **linear** if it also preserves the lift, that is,

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \lambda \downarrow & & \downarrow \lambda' \\ T(E) & \xrightarrow[T(f)]{} & T(E') \end{array}$$

  commutes.

Note: this does correspond to the ordinary definition of linear morphisms between vector bundles in the canonical example.

(Vector-valued) Differential forms

Definition

If M is an object of $\mathbb{X}$ and $q : E \longrightarrow M$ a differential bundle, a E-**valued differential** n-**form on** M consists of a map

$$\omega : T_n M \longrightarrow E$$

which is "linear in each variable" and alternating.

In the case when the differential bundle is of the form $\pi_2 : A \times M \longrightarrow M$ for some differential object A, these are ordinary differential forms - in particular in the canonical example, when $A = \mathbb{R}$.

Connections (intuitively)

Intuitive idea: can "move tangent vectors between different tangent spaces". Moving a tangent vector around a closed curve measures the "curvature" of the space. Connections have been expressed in many different ways:

- as a "horizontal subspace";
- as a "connection map";
- as a notion of "parallel tranport";
- as a "covariant derivative".

Quoting Michael Spivak:

> *"I personally feel that the next person to propose a new definition of a connection should be summarily executed."*

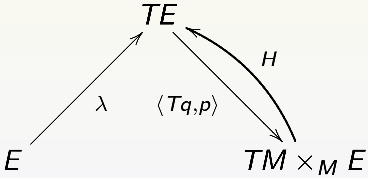
Two fundamental maps

A differential bundle has two key maps involving TE whose composite is the zero map:

$$E \xrightarrow{\lambda} TE \xrightarrow{\langle Tq, p \rangle} TM \times_M E$$

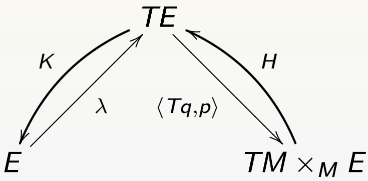
Horizontal lift

A **connection** consists of a linear section of H of $\langle Tq, p \rangle$ called the **horizontal lift**...



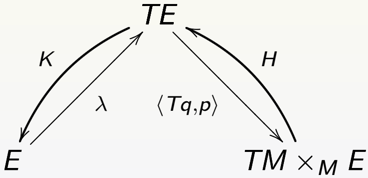
Connector

which in addition has a linear retraction K of λ called the **connector**:



Connection definition

that satisfies the equations $KH = 0$ and $(\lambda K \oplus 0p) + H\langle Tq, p\rangle = 1_{TE}$.



Simple example

Any differential object A is a differential bundle over 1 and these have a canonical connection given by:

- $K : TA \longrightarrow A$ by $K(v, a) := v$ and
- $H : A \longrightarrow TA$ by $H(a) := (0, a)$.

K from H and vice versa

Suppose $(\mathbb{X}, \mathbb{T})$ is a tangent category with negation and (q, λ) is a differential bundle.

Proposition

If H is a linear section of $\langle T(q), p \rangle$, then q *can be given the structure of a connection with horizontal lift H.*

Proposition

If K is a linear retract of λ, and q *has at least one section J of $\langle T(q), p \rangle$, then* q *can be given the structure of a connection with connector K.*

Flat connections

The definition of a connection being flat in the literature is quite complicated, but by using the map c we can make a very simple definition:

Definition

Say that a connection is **flat** if $cT(K)K = T(K)K$.

One can show this is equivalent to the standard definition (involving curvature) in the canonical example.

Affine and torsion-free connections

Torsion-free connections are connections on the tangent bundle for which the movement of tangent vectors does not "twist". Again there is a simple definition of this in our setting:

Definition

When the connection is on a tangent bundle $p : TM \longrightarrow M$, the connection is called **affine**. Say an affine connection is **torsion-free** if $cK = K$.

This again is again equivalent to the usual definition (involving the Lie bracket) in the canonical example.

Conclusions

- Most categories related to differential or algebraic geometry are tangent categories.
- The following are well-defined notions in any tangent category: vector fields, the Lie bracket, "vector" spaces and bundles, differential forms, and connections.
- The definitions of differential object and bundle shed light on the nature of vector spaces and bundles in differential geometry.
- The definition of connections, as well as their properties of being torsion-free and affine, shed light on connections in differential geometry.

References

References:

- Cockett, R. and Cruttwell, G. Differential structure, tangent structure, and SDG. *Applied Categorical Structures*, Vol. 22 (2), pg. 331–417, 2014.
- Rosický, J. Abstract tangent functors. *Diagrammes*, 12, Exp. No. 3, 1984.