A simplicial framework for de Rham cohomology in a tangent category

Geoff Cruttwell
(joint work with Rory Lucyshyn-Wright)
Mount Allison University

Category Theory 2016
Halifax, Canada, August 13, 2016

Overview

- Tangent categories provide an abstract framework to develop many concepts in differential geometry.
- Many key concepts and results from differential geometry have already been developed in this framework (Lie bracket, vector bundles, connections). But differential forms and de Rham cohomology have proven elusive.

Overview

- Tangent categories provide an abstract framework to develop many concepts in differential geometry.
- Many key concepts and results from differential geometry have already been developed in this framework (Lie bracket, vector bundles, connections). But differential forms and de Rham cohomology have proven elusive.
- In this talk we'll look at variants of the notion of differential form in tangent categories.
- In particular, we'll look at *sector forms*, and show that they have very rich structure. Our results about this structure appear to be new, even in ordinary differential geometry.
- From the sector forms, we'll get a definition of de Rham cohomology in a tangent category as a simple corollary.

Tangent category definition

Definition (Rosický 1984, modified Cockett/Cruttwell 2013)

A **tangent category** consists of a category $\mathbb{X}$ with:

- an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of $p_M : TM \longrightarrow M$ along itself exists (and is preserved by each T^m), call this pullback $T_n M$;
- for each $M \in \mathbb{X}$, $p_M : TM \longrightarrow M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformations $+ : T_2 \longrightarrow T$, $0 : 1_{\mathbb{X}} \longrightarrow T$;

(TM represents the "tangent bundle" of an object M.)

Tangent category definition (continued)

Definition

- (canonical flip) there is a natural transformation $c : T^2 \longrightarrow T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \longrightarrow T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for ℓ and c;
- (universality of vertical lift) "an element of T^2M which has $T(p) = 0$ is uniquely given by an element of T_2M".

Examples

(i) Finite dimensional smooth manifolds with the usual tangent bundle.

(ii) Convenient manifolds with the kinematic tangent bundle.

(iii) Any Cartesian differential category (includes all Fermat theories by a result of MacAdam).

(iv) The infinitesimally linear objects in a model of synthetic differential geometry (SDG).

(v) Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.

(vi) The category of C^{∞}-rings.

Note: Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.

Differential objects

Definition

A **differential object** in a tangent category consists of a commutative monoid E with a map $\hat{p} : TE \longrightarrow E$ such that

$$E \xleftarrow{\hat{p}} TE \xrightarrow{p_E} E$$

is a product diagram, and such that $\hat{p}$ satisfies various coherences with the tangent structure.

Examples:

- $\mathbb{R}^n$'s in the category of smooth manifolds.
- Convenient vector spaces in the category of convenient manifolds.
- Euclidean R-modules in models of SDG.

Differential objects II

- Differential objects also have a map

$$\lambda : E \longrightarrow TE$$

which will be useful when defining “linear” maps to these objects.

- If E is a differential object, any map

$$X \xrightarrow{f} E$$

has an associated “derivative” $D(f) : TX \longrightarrow E$ given by

$$TX \xrightarrow{Tf} TE \xrightarrow{\hat{p}} E$$

Classical differential forms

- The classical notion of differential n-form on a smooth manifold M is a smooth map
$$T_nM \xrightarrow{\omega} \mathbb{R}$$
which is multilinear and alternating (switching two of the inputs gives the negative).
- In a tangent category, we have the objects T_nM, can replace $\mathbb{R}$ with a differential object E, and give a suitable definition of multilinear and alternating to get "classical" differential forms as multilinear alternating maps
$$T_nM \xrightarrow{\omega} E$$

Derivatives of classical differential forms

- But the exterior derivative of a classical form ω is problematic.
- Classically, the exterior derivative is defined locally (not possible in an arbitrary tangent category!) by an alternating sum of various derivatives of ω.
- In a tangent category, if we have a classical form

$$T_n(M) \xrightarrow{\omega} E$$

then its derivative is

$$T(T_nM) \xrightarrow{D(\omega)} E$$

which is not the right type.

- An arbitrary M does not have a canonical choice of map

$$T_{n+1}(M) \to T(T_n(M))$$

to get a classical $(n+1)$-form.

Singular forms

- In SDG, one instead considers *singular* forms: maps

$$T^n(M) \xrightarrow{\omega} E$$

suitably multilinear and alternating.

- In smooth manifolds, giving such a map is equivalent to giving a classical form (!).
- One can similarly define singular forms in tangent categories,and define an appropriate exterior derivative for such singular forms in a tangent category, as the derivative of ω

$$T^{n+1}(M) \xrightarrow{D(\omega)} E$$

has the correct type (the exterior derivative is then defined as an alternating sum of permutations of this derivative).

Sector forms

- When calculating with singular forms, it becomes natural to consider maps

$$T^n(M) \xrightarrow{\omega} E$$

which are merely multilinear (not necessarily alternating).

- These are known as "sector forms", and have been investigated only briefly in differential geometry in a book by J.E. White.
- These will be the main object of interest for us.

Comparison of forms

For comparison:

- $T_n(M)$: n (first-order) tangent vectors on M.
- $T^n(M)$: nth order tangent vector on M.
- There is a canonical map $T^n(M) \longrightarrow T_n(M)$.
- Thus sector forms generalize classical forms, singular forms, and covariant tensors:

	alternating	not alternating
domain T_n	differential form	covariant tensor
domain T^n	singular form	sector form

Definition of sector forms in a tangent category

Definition

A **sector** n**-form** on M with values in E is a morphism $\omega : T^n M \to E$ such that for each $i \in \{1, \ldots, n\}$, ω is *linear in the* i*th variable*; that is, the following diagram commutes:

$$\begin{array}{ccc} T^n M & \xrightarrow{\ \omega\ } & E \\ {\scriptstyle a_i^n}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\ T^{n+1} M & \xrightarrow[T(\omega)]{} & TE \end{array}$$

(where $a_1^n = \ell$, $a_2^n = cT(\ell)$, $a_3^n = cT(c)T^2(\ell)$, etc.)

The set of sector n forms on M with values in E will be denoted by $\Psi_n(M; E)$; we will often abbreviate this to $\Psi_n(M)$.

Fundamental derivative of a sector form

- There is an operation

$$\delta_1 : \Psi_n(M) \longrightarrow \Psi_{n+1}(M)$$

given by sending a sector n-form

$$\omega : T^n M \longrightarrow E$$

to the sector $(n+1)$-form

$$D(\omega) : T^{n+1} M \longrightarrow E$$

- **Note**: even if ω is alternating, $\delta_1(\omega) := D(\omega)$ need not be.
- But there are actually n other related "derivatives"...

Symmetry operations

- For any $n \geq 2$, pre-composing a sector n-form ω with the canonical flip again gives an n-form:

$$T^n M \xrightarrow{c_{T^{n-2}M}} T^n M \xrightarrow{\omega} E$$

giving an operation

$$\sigma_1 : \Psi_n M \longrightarrow \Psi_n M$$

- And for higher n, pre-composing with $T(c_{T^{n-3}M}), T^2(c_{T^{n-4}M})$, etc. gives $n-1$ different *symmetry* operations

$$\sigma_1, \sigma_2, \ldots \sigma_{n-1} : \Psi_n M \longrightarrow \Psi_n M$$

Derivative/coface operations

- By post-composing the fundamental derivative

$$\delta_1 : \Psi_n(M) \longrightarrow \Psi_{n+1}(M)$$

with the first symmetry

$$\sigma_1 : \Psi_{n+1}(M) \longrightarrow \Psi_{n+1}(M)$$

we get a new "derivative"

$$\delta_2 : \Psi_n(M) \longrightarrow \Psi_{n+1}(M)$$

- Post-composing this with σ_2 gives δ_3, then δ_4, etc...continuing in this way we get $(n+1)$ total ways to get an $(n+1)$-form from an n-form, notated as

$$\delta_1, \delta_2, \delta_3, \ldots \delta_{n+1} : \Psi_n M \longrightarrow \Psi_{n+1} M$$

which we refer to as the *co-face* operations.

Codegeneracy operations

- For a sector n-form $\omega : T^n M \longrightarrow E$, pre-composing with the lift ℓ gives an $(n-1)$-form:

$$T^{n-1}M \xrightarrow{\ell_{T^{n-2}M}} T^n M \xrightarrow{\omega} E$$

giving an operation

$$\varepsilon_1 : \Psi_n M \longrightarrow \Psi_{n-1} M$$

- Similarly, for higher n, pre-composing with $T(\ell_{T^{n-3}M})$, $T^2(\ell_{T^{n-4}M})$, etc. gives $n-1$ different *codegeneracy* operations

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1} : \Psi_n M \longrightarrow \Psi_{n-1} M$$

Symmetric cosimpicial objects

Definition (Grandis/Barr)

An (augmented) **symmetric cosimpicial object** in a category $\mathbb{X}$ consists of a sequence of objects

$$C_0, C_1, C_2, \ldots, C_n, \ldots$$

with, for each n, maps

$$\delta_i^n : C_n \longrightarrow C_{n+1} \text{ for each } i = 1 \ldots n+1; \textbf{ (Cofaces)}$$

$$\varepsilon_i^n : C_n \longrightarrow C_{n-1} \text{ for each } i = 1 \ldots n-1; \textbf{ (Codegeneracies)}$$

$$\sigma_i^n : C_n \longrightarrow C_n \text{ for each } i = 1 \ldots n-1 \textbf{ (Symmetries)}$$

satisfying 15 equations relating these maps, for example, for $i < j$,

$$\varepsilon_j \delta_i = \delta_i \epsilon_{j-1}.$$

Such an object is equivalent to giving a functor

$$C : \textbf{finCard} \longrightarrow \mathbb{X}.$$

Main result

Theorem

Let $\mathbb{X}$ be a tangent category with a differential object E.

- *Each object M has an associated symmetric cosimpicial monoid $\Psi(M)$, where $\Psi_n(M)$ is the set of of sector n-forms, and cofaces, codegeneracies, and symmetries are as described previously.*
- *This assignment is contravariantly functorial.*

Main result

Theorem

Let $\mathbb{X}$ be a tangent category with a differential object E.

- *Each object M has an associated symmetric cosimpicial monoid $\Psi(M)$, where $\Psi_n(M)$ is the set of of sector n-forms, and cofaces, codegeneracies, and symmetries are as described previously.*
- *This assignment is contravariantly functorial.*

Corollary

For each function $f : n \longrightarrow m$ between finite cardinals there is an associated map between sector forms

$$\Psi_f : \Psi_n(M) \longrightarrow \Psi_m(M).$$

These appear to be new results in the category of smooth manifolds.

Corollary: Cochain complex of sector forms

- Now suppose that the differential object E is “subtractive”; that is, it's underlying monoid is in fact a group.
- In this case, each $\Psi(M)$ is actually a symmetric cosimpicial group.
- Any cosimpicial group Ψ has an associated map $\delta^n : \Psi_n \longrightarrow \Psi_{n+1}$ given by

$$\partial^n(\omega) := \sum_{i=1}^{n+1}(-1)^{i-1}\delta_i^n(\omega)$$

which has the property that $\delta^{n+1}(\delta^n(\omega)) = 0$.

Corollary

If E is subtractive, each $\Psi(M; E)$ can be given the structure of a cochain complex.

This also appears to be a new result for smooth manifolds.

Corollary: Cochain complex of singular forms

- Recall that singular forms are alternating sector forms.
- It is easy to show that the above operation ∂ restricts to singular forms.

Corollary

If E is subtractive, the singular forms on M with values in E have the structure of a cochain complex.

In the category of smooth manifolds, this cochain complex is the de Rham complex.

Conclusions

- Sector forms in tangent categories have a very rich structure which has not previously been fully described, even in the canonical category of smooth manifolds.
- As a consequence, tangent categories support a notion of generalization of de Rham cohomology (and in fact possess a possibly distinct cohomology of sector forms).

Conclusions

- Sector forms in tangent categories have a very rich structure which has not previously been fully described, even in the canonical category of smooth manifolds.
- As a consequence, tangent categories support a notion of generalization of de Rham cohomology (and in fact possess a possibly distinct cohomology of sector forms).
- (J. E. White) If $g : T_2 M \longrightarrow \mathbb{R}$ is a Pseudo-Riemannian metric on M (in particular, a covariant 2-tensor) quantities like the *cycle*

$$\delta_1 g + \delta_2 g - \delta_3 g,$$

and *balance*

$$\delta_1 g - \delta_2 g$$

of g are sector forms which are not themselves tensors; thus general results about sector forms may further understanding of such invariants.

References

- Cruttwell, G. and Lucyshyn-Wright, R. **A simplicial framework for de Rham cohomology in tangent categories**. Submitted; available at arXiv:1606.09080.
- Cockett, R. and Cruttwell, G. **Differential structure, tangent structure, and SDG**. *Applied Categorical Structures*, Vol. 22 (2), pg. 331–417, 2014.
- Rosický, J. **Abstract tangent functors**. *Diagrammes*, 12, Exp. No. 3, 1984.
- White, J. **The method of iterated tangents with applications in local Riemannian geometry**, Pitman publishing, 1982.