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Cartesian Differential Categories Revisited

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We revisit the definition of Cartesian differential categories, showing that a slightly more general version is useful for a number of reasons. As one application, we show that these general differential categories are comonadic over categories with finite products, so that every category with finite products has an associated cofree differential category. We also work out the corresponding results when the categories involved have restriction structure, and show that these categories are closed under splitting restriction idempotents.

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1. Introduction

Cartesian differential categories (Blute et. al. 2009) were developed as an axiomatization of the essential properties of the derivative. The standard example is differentiation of smooth functions between Cartesian spaces, but there are many other examples, such as differentiation of polynomials, differentiation of smooth functions between convenient vector spaces (Blute et. al. 2011), and differentiation of data types. With an additional axiom, the definition gives the categorical semantics for the differential lambda calculus of (Erhard and Regnier 2003), as described in (Manzonetto 2012). Every category with an abstract "tangent functor" (Rosický 1984) has an associated Cartesian differential category (Cockett and Cruttwell 2013). For example, any model of synthetic differential

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geometry (Kock 2006) has an associated Cartesian differential category. Every (monoidal) differential category (Blute et. al. 2006) has an associated Cartesian differential category, and (Laird et. al. 2013) shows how one can construct a monoidal differential category from any symmetric monoidal category. Finally, the paper (Cockett and Seely 2011) demonstrated the surprising result that there are (co)free instances of Cartesian differential categories.

However, examined more closely, there are a number of problems with the definition of a Cartesian differential category that all point to a similar root defect. The first objection is philosophical. In a Cartesian differential category, every map $f : X \longrightarrow Y$ has an associated differential map $D(f) : X \times X \longrightarrow Y$. However, the two X's in the domain of $D(f)$ play different roles. In the canonical example of the category of smooth maps between finite-dimensional vector spaces, $D(f)$ is the Jacobian, evaluated at the second X, then applied in the direction of the first X. In other words, we think of the first X as consisting of vectors, and the second X as consisting of points. The dual nature of X is not reflected in the definition, and leads one to consider whether it may be possible for the two X's to in fact be different objects.

A second objection occurs when one inspects the comonadicity of Cartesian differential categories. In (Cockett and Seely 2011), the authors showed that there was a comonad on left additive Cartesian categories for which certain coalgebras were Cartesian differential categories. This leads one to wonder what the general coalgebras may be (the authors themselves note this, saying "The more general construction...seems actually to be more natural, and this is an indication that the construction has more general forms which we shall not explore here" (pg. 397–398)). Again, the more general coalgebras point to a definition in which the domain of the derivative should be two different objects.

The final objection occurs when one combines differential categories with restriction categories, as is done in (Cockett et. al 2011). One of the most basic operations on any restriction category is to split the restriction idempotents. Unfortunately, differential restriction categories are not closed under this operation. Again, the problem is resolved by allowing the two elements of the domain of $D(f)$ to be seperate, so that, for example, while the derivative may only be evaluated in some open set $U \subseteq \mathbb{R}^n$, the vector along which it is taken is any vector in $\mathbb{R}^n$.

With these considerations in mind, we reformulate the definition of Cartesian differential categories (and later, differential restriction categories). In the new definition, not every object need have the structure of a commutative monoid. Instead, to each object X there is an assigned commutative monoid $L(X) = (L_0(X), +_X, 0_X)$ which we think of as the "object of vectors" associated to the "object of points" X (naturally, one of the axioms for this operation is $L(L_0(X)) = L(X)$). The derivative of a map $f : X \longrightarrow Y$ is then a map $L_0(X) \times X \longrightarrow L_0(Y)$ satisfying almost identical axioms to those for Cartesian differential categories.

Not only does this more general version solve all the problems mentioned above, it also reveals a striking new property. In the original version, Cartesian differential categories were comonadic over Cartesian left additive categories. In the new version, Cartesian differential categories are comonadic over Cartesian categories (that is, categories with finite products). As nearly every naturally-occuring category has finite products, this shows that there are a vast number of Cartesian differential categories. This is a remarkable result, considering the intricacy of the axioms, and underlines the importance of the constructions in (Cockett and Seely 2011).

The paper is laid out as follows. We begin by giving our generalized definition of Cartesian differential categories, then show how they are the coalgebras for a slightly modified version of the Faà di bruno comonad of (Cockett and Seely 2011). Fortunately, most of the work has been done in (Cockett and Seely 2011); only one small modification to how one of the differential axioms is arrived at is required. In addition, we determine the nature of the linear maps in these cofree examples.

Following this, we work out the corresponding restriction versions. Here, the proofs require more detail: while several aspects of the proofs remain as they were in (Cockett and Seely 2011), care is required to ensure that all works well with the restriction structure. In fact, it should be noted that the Faà di bruno construction has not been given even for non-generalized differential restriction categories, so this section in fact generalizes (Cockett and Seely 2011) in two different ways. We first give the generalized version of the differential restriction categories of (Cockett et. al 2011), and show that unlike their ordinary counterparts, they are closed when we split the restriction idempotents. Following this, we give a restriction version of the Faà di bruno comonad. The end result is again striking: every Cartesian restriction category has an associated (generalized) differential restriction category.

It should be noted that since every category is in fact a restriction category with a trivial restriction structure, the results of section 2 can be seen as a corollary to the results of section 3. However, we have chosen to present the total case first, so that the reader can understand the constructions at the ordinary categorical level before getting bogged down in the additional details required when restriction structure is present.

The work done in this paper leads to an obvious next step: determine the nature of these cofree Cartesian differential categories, and understand how they may be used. In particular, it may be worth understanding the associated tangent structure (Cockett and Cruttwell 2013) of these examples.

2. Cartesian differential categories revisited

We begin by generalizing the central definition of (Blute et. al. 2009). As noted in the introduction, the main point of generalization is to allow examples where not all objects need have the structure of a commutative monoid. Instead, each object X has an associated commutative monoid satisfying two axioms; one thinks of this object as the "vectors" associated to the object X. The derivative of a map with domain X then has domain taking values in the product of the X with its object of vectors.

Before the definition, we note a few conventions we will use thoughout the paper. Following (Blute et. al. 2009) and (Cockett et. al 2011), we write composition diagrammatically, so that first applying f, then applying g, is written as fg. If we have a monoid $(A, +_A, 0_A)$ and maps $f, g : X \longrightarrow A$, we will use $f + g$ to denote $\langle f, g\rangle +_A$ and $0 : X \longrightarrow A$ to denote the map $!0_A$. A **Cartesian category** will mean a category with chosen finite products.

Definition 2.1. A **generalized Cartesian differential category** consists of a Cartesian category $\mathbb{X}$ with:

— for each object X, a commutative monoid $L(X) = (L_0(X), +_X, 0_X)$, satisfying

$$L(L_0(X)) = L(X) \text{ and } L(X \times Y) = L(X) \times L(Y),$$

— for each map $f : X \longrightarrow Y$, a map $D(f) : L_0(X) \times X \longrightarrow L_0(Y)$ such that:

[CD.1] $D(+_X) = \pi_0 +_X, D(0_X) = \pi_0 0_X$,

[CD.2] $\langle a + b, c\rangle D(f) = \langle a, c\rangle D(f) + \langle b, c\rangle D(f)$ and $\langle 0, a\rangle D(f) = 0$;

[CD.3] $D(\pi_0) = \pi_0\pi_0$, $D(\pi_1) = \pi_0\pi_1$, and $D(1) = \pi_0$;

[CD.4] $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$;

[CD.5] $D(fg) = \langle D(f), \pi_1 f\rangle D(g)$;

[CD.6] $\langle\langle a, 0\rangle, \langle c, d\rangle\rangle D^2(f) = \langle a, d\rangle D(f)$;

[CD.7] $\langle\langle 0, b\rangle, \langle c, d\rangle\rangle D^2(f) = \langle\langle 0, c\rangle, \langle b, d\rangle\rangle D^2(f)$.

It may be helpful to give the reader some intuition for these axioms. **[CD.1]** says that differentiation preserves addition. **[CD.2]** says that the derivative is additive in its first variable. **[CD.3]** and **[CD.4]** demand that differentiation is compatible with the product structure of the category. **[CD.5]** is the chain rule. **[CD.6]** is a formulation of the fact that the derivative is linear in its first variable, and **[CD.7]** represents the symmetry of second partial derivatives.

Note that only **[CD.1]** has a slightly different form than given for Cartesian differential categories. In fact, however, the definition given here is an equivalent version of the first axiom for Cartesian differential categories (Blute et. al. 2009):

Lemma 2.2. In the presence of the other axioms, **[CD.1]** is equivalent to asking that for maps $f, g : X \longrightarrow L(Y)$,

$$D(f + g) = D(f) + D(g) \text{ and } D(0) = 0.$$

Proof. Suppose we know $D(f + g) = D(f) + D(g)$ and $D(0) = 0$. Note that $+_X = \pi_0 + \pi_1$, so we have

$$D(+_X) = D(\pi_0 + \pi_1) = D(\pi_0) + D(\pi_1) = \pi_0\pi_0 + \pi_0\pi_1 = \pi_0(+_X),$$

and similarly for 0.

Conversely, suppose we know that **[CD.1]** is satisfied. Consider:

$$D(f + g) = D(\langle f, g\rangle +_X) = \langle D(\langle f, g\rangle), \pi_1\langle f, g\rangle\rangle D(+_X) =$$

$$\langle\langle Df, Dg\rangle, \pi_1\langle f, g\rangle\rangle\pi_0 +_X = \langle Df, Dg\rangle +_X = Df + Dg,$$

and similarly for the preservation of 0. □

The version of **[CD.1]** we give in our axiomatization above is perhaps slightly more natural than the version given in the lemma, as it does not require us to quantify over all $f, g : X \longrightarrow L(Y)$.

Lemma 2.2 then tells us that:

Remark 2.3. Any Cartesian differential category is a generalized Cartesian differential category, with $L(X) := (X, \pi_0 + \pi_1, 0)$.

For example, the category of finite dimensional vector spaces and smooth maps between them, or the category of convenient vector spaces and smooth maps between them (Blute et. al. 2011) are examples of generalized Cartesian differential categories.

However, importantly, there are examples of generalized Cartesian differential categories which are not (ordinary) Cartesian differential categories. Applying corollory 3.9 gives us the following non-trivial generalized example:

Example 2.4. The category with objects $U \subseteq \mathbb{R}^n$ and smooth maps

$$f : (U \subseteq \mathbb{R}^n) \longrightarrow (V \subseteq \mathbb{R}^m)$$

forms a generalized Cartesian differential category, with the Jacobian as the derivative and

$$L_0(U \subseteq R^n) = R^n.$$

Note that this is a true generalized example, as the Jacobian of a map

$$U \subseteq \mathbb{R}^n \xrightarrow{f} V \subseteq \mathbb{R}^m$$

takes as input any vector in $\mathbb{R}^n$ (not merely those in U) and could return any vector in $\mathbb{R}^m$ (not merely one in V). One can construct a similar generalized Cartesian differential category in which the objects are open subsets of convenient vector spaces, with maps smooth maps defined on those open subsets.

It is also important to note that generalizing the definition in this way also allows for trivial examples:

Example 2.5. If $\mathbb{X}$ is a Cartesian category, then defining

$$L(X) := 1 \text{ and } D(f) := !,$$

for any object X and map $f : X \longrightarrow Y$ gives $\mathbb{X}$ the structure of a generalized Cartesian differential category.

As a source of further examples (which are not Cartesian differential categories), in the next section, we shall see that from any category with finite products we can produce a generalized Cartesian differential category.

2.1. *The Faà di Bruno comonad and its coalgebras*

In this section, we generalize the Faà di Bruno comonad of (Cockett and Seely 2011) to generalized Cartesian differential categories. As we shall see, we can construct a comonad that is almost identical to that in (Cockett and Seely 2011); however, in our version, the comonad is on all Cartesian categories, as opposed to left additive Cartesian categories (that is, Cartesian categories we a certain addition operation). Thus, as a result, any Cartesian category has an associated cofree generalized Cartesian differential category.

Given a Cartesian category $\mathbb{X}$, the idea of the Faà di Bruno comonad is to take a category $\mathbb{X}$, and (i) replace the objects with pairs (A, X), where A is a commutative monoid of $\mathbb{X}$, and X an arbitrary object of $\mathbb{X}$, and (ii) replace the arrows with sequences of arrows. One thinks of the pair (A, X) as "an object X, together with its object of vectors A". However, it is important to note that we will require no relationship between A and X. An arrow $f : (A, X) \longrightarrow (B, Y)$ in this category is a sequence of arrows $(f_*, f_1, f_2, \ldots f_n \ldots)$, with $f_* : X \longrightarrow Y$ an arrow between the base objects, $f_1 : A \times X \longrightarrow B$ taking a vector and a point and returning a vector, and for $n > 1$:

$$A^n \times X \xrightarrow{f_n} B,$$

so that f_n takes n vectors of A, a point of X, and returns a vector of B. Intuitively, one should think of f_* as the basic arrow, f_1 as its first derivative, f_2 as its second derivative, etc, so we will also require that f_n be additive in its first n variables and symmetric in those variables. However, just like we require no relationship between the "vectors" A and the "points" X of an object (A, X), we will require no relationship between the different f_n's.

The "differential" part of this category will appear in how we compose the arrows. Given arrows

$$(A, X) \xrightarrow{f} (B, Y) \xrightarrow{g} (C, Z),$$

there is an obvious composite for the base maps: $(fg)_* := f_* g_*$. Given the types of f_1 and g_1, there is not an obvious definition for $(fg)_1$. However, if we are thinking of f_1 and g_1 as derivatives, then there is an obvious choice: we will define $(fg)_1$ by the chain rule. That is, we will define $(fg)_1 : A \times X \longrightarrow C$ by the composite

$$A \times X \xrightarrow{\langle f_1, \pi_1 f_0 \rangle} B \times Y \xrightarrow{g_1} Z.$$

Comparing this with [**CD.5**], one can see that this truly is the chain rule, again thinking of f_1 and g_1 as the first derivatives of f and g respectively.

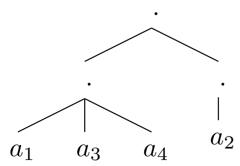
The difficulty is then determining how to compose the higher "derivatives" of f and g. Fortunately, as noted in (Cockett and Seely 2011), this has already been done by Faà di Bruno himself, who worked out the combinatorics of the nth derivative of a composite function.

As described in (Cockett and Seely 2011), the easiest way to view the combinatorics of the higher-order chain rule is via trees. Given a symmetric tree τ of height 2 and width n on the variables $\{a_1, a_2, \ldots a_n\}$, we define

$$(f \star g)_\tau : A^r \times X \longrightarrow C$$

by substituting all the level one nodes of arity i of τ with f_i, and the single node of τ at level two with the function g_j, where j is the number of branches of that single node.

For example, for the tree



and an element

$$Z \xrightarrow{z=\langle a_1,a_2,a_3,a_4,x\rangle} A^4 \times X,$$

$(fg)_\tau$ applied to z is

$$\langle\langle a_2, x\rangle f_1, \langle a_1, a_3, a_4, x\rangle f_3, f_*(x)\rangle g_2.$$

We then define $(fg)_n$ by summing the elements $(fg)_\tau$ over all τ a symmetric tree of height 2 and length n.

Following [**CD.3**], we expect the identity maps for this category to have "first derivative" π_0, and higher derivatives 0.

We are now in a position to define the Faà di Bruno comonad.

Definition 2.6. Let **cartCat** denote the category whose objects are Cartesian categories, and whose arrows are functors which preserve the specified products exactly. Given objects A, B in a Cartesian category, define $\mathrm{ex} : A \times B \times A \times B \longrightarrow A \times A \times B \times B$ to be the obvious "switch" map.

Proposition 2.7. There is an endofunctor on **cartCat**, **Faà**, with **Faà**$(\mathbb{X})$ having:

— objects pairs $((A, +, 0), X)$ with $(A, +, 0)$ a commutative monoid in $\mathbb{X}$, and X an object of $\mathbb{X}$;

— a morphism from $((A, +_A, 0_A), X)$ to $((B, +_B, 0_B), Y)$ consists of an infinite sequence of maps $(f_*, f_1, f_2, \ldots)$ with $f_* : X \longrightarrow Y$ simply a map in X, and for each n, $f_n : A \times A \times \ldots \times A \times X \longrightarrow B$ is a map in $\mathbb{X}$ that is additive and symmetric in its first n variables,

— composition is as above, while the identity on (A, X) is the map

$$(1_X, \pi_0, 0, 0, \ldots),$$

— with the product

$$((A, +_A, 0_A), X) \times ((B, +_B, 0_B), Y) := ((A \times B, \mathrm{ex}(+_A \times +_B), 0_A \times 0_B), X \times Y)$$

and projections

$$\pi_{(A,X)} := (\pi_X, \pi_0\pi_A, 0, 0, \ldots), \pi_{(B,Y)} := (\pi_Y, \pi_0\pi_B, 0, 0, \ldots),$$

and, given a Cartesian functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$, **Faà**$(F)$ has the obvious action:

— **Faà**$(F)((A, +, 0), X) = ((FA, F+, F0), FX)$;

— **Faà**$(F)(f_*, f_1, f_2, \ldots) = (F(f_*), F(f_1), F(f_2), \ldots)$,

which is well-defined since F preserves the specified products.

Proof. The proof is identical to that of lemma 2.2.1 in (Cockett and Seely 2011). $\square$

Note that there are two key differences between the monad presented here and the one presented in (Cockett and Seely 2011). In the version there, the base categories already have all objects additive, and the action of the monad on a category $\mathbb{X}$ has objects pairs (A, A). In our version, the base categories are mere categories with products, while the action on a category $\mathbb{X}$ gives a category with objects pairs (A, X), with A a commutative monoid.

In addition, in this more generalized setting, there is a natural comparison between this endofunctor and the commutative monoid endofunctor, which does not appear at the level considered in (Cockett and Seely 2011).

Definition 2.8. let **cMon** denote the endofunctor on **cartCat** which sends a category $\mathbb{X}$ to its category of commutative monoids and additive maps between them (with its obvious product structure).

Proposition 2.9. There is a natural transformation $\lambda : \mathbf{cMon}(\mathbb{X}) \longrightarrow \mathbf{Fa\grave{a}}(\mathbb{X})$, which maps

$$(A, +, 0) \mapsto ((A, +, 0), A)$$

and

$$A \xrightarrow{f} B \mapsto (f, \pi_0 f, 0, 0, \ldots).$$

Proof. $\lambda_{\mathbb{X}}(f)$ is a valid map in $\mathbf{Fa\grave{a}}(\mathbb{X})$ since f is additive. For each $\mathbb{X}$, $\lambda_{\mathbb{X}}$ is a functor since $1_{((A,+,0),A)} = (1, \pi_0, 0, \ldots)$ and

$$\begin{aligned}
\lambda_{\mathbb{X}}(f)\lambda_{\mathbb{X}}(g) &= (f, \pi_0 f, 0, \ldots)(g, \pi_0 g, 0, \ldots) \\
&= (fg, \langle \pi_0 f, \pi_1 f \rangle \pi_0 g, 0, \ldots) \\
&= (fg, \pi_0 fg, 0, \ldots) \\
&= \lambda_{\mathbb{X}}(fg).
\end{aligned}$$

It preserves products since the projections in $\mathbf{Fa\grave{a}}(\mathbb{X})$ are $\lambda_{\mathbb{X}}(\pi_0)$, $\lambda_{\mathbb{X}}(\pi_1)$. For naturality, for a product-preserving functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ we need the diagram

$$\begin{array}{ccc}
\mathbf{cMon}(\mathbb{X}) & \xrightarrow{\lambda_{\mathbb{X}}} & \mathbf{Fa\grave{a}}(\mathbb{X}) \\
{\scriptstyle \mathbf{cMon}(F)} \downarrow & & \downarrow {\scriptstyle \mathbf{Fa\grave{a}}(F)} \\
\mathbf{cMon}(\mathbb{Y}) & \xrightarrow[\lambda_{\mathbb{Y}}]{} & \mathbf{Fa\grave{a}}(\mathbb{Y})
\end{array}$$

to commute. It is easy to see that on objects these two composite functors are equal, while for an addition-and-0-preserving map $f : (A, +_A, 0_A) \longrightarrow (B, +_B, 0_B) \in \mathbb{X}$,

$$\lambda_{\mathbb{Y}}(\mathbf{cMon}(F)(f)) = \lambda_{\mathbb{Y}}(F(f)) = (F(f), \pi_0 F(f), F(0_B), \ldots)$$

(since the 0 in the codomain is $F(0_B)$), while

$$\mathbf{Fa\grave{a}}(F)(\lambda_{\mathbb{X}}(f)) = \mathbf{Fa\grave{a}}(F)(f, \pi_0 f, 0_B, \ldots) = (F(f), \pi_0 F(f), F(0_B), \ldots)$$

since F preserves the specified products. So λ is natural, as required. $\square$

Corollary 2.10. If $(A, +, e)$ is a commutative monoid in $\mathbb{X}$, then

$$(((A, +, 0), A), (+, \pi_0+, 0, \ldots), (0, \pi_0 0, 0, \ldots))$$

is a commutative monoid in $\mathbf{Fa\grave{a}}(\mathbb{X})$.

Proof. Since $(A, +, 0)$ is commutative, $(A, +, 0)$ is a commutative monoid in $\mathbf{cMon}(\mathbb{X})$, and so gets sent by $\lambda_{\mathbb{X}}$ to a commutative monoid in $\mathbf{Fa\grave{a}}(\mathbb{X})$. □

We can now describe the comonad structure of our generalized version of **Faà**, again generalizing the work from (Cockett and Seely 2011). There is an obvious co-unit ϵ : $\mathbf{Fa\grave{a}}(\mathbb{X}) \longrightarrow \mathbb{X}$ which maps (A, X) to X and $f : (A, X) \longrightarrow (B, Y)$ to $f_* : X \longrightarrow Y$. The co-multiplication δ is more complicated. On arrows it should be thought of giving an abstract "derivative" of a map $f = (f_*, f_1, f_2, \ldots) : (A, X) \longrightarrow (B, Y)$. On objects, it will map (A, X) to $((A, A), (A, X))$. For a map $f = (f_*, f_1, f_2, \ldots) : (A, X) \longrightarrow (B, Y)$ in $\mathbf{Fa\grave{a}}(\mathbb{X})$, note that $\delta(f) : ((A, A), (A, X)) \longrightarrow ((B, B), (B, Y))$ will be a map in $\mathbf{Fa\grave{a}}^2(\mathbb{X})$, and so it will be a sequence of maps, each of which is itself a sequence of maps. There is a natural choice for $\delta(f)_*$: as its type should be a map $(A, X) \longrightarrow (B, Y)$ in $\mathbf{Fa\grave{a}}(X)$, we can simply define $\delta(f)_*$ as f.

For $n \geq 1$, there is also a natural choice for $(\delta(f)_n)_*$: its type is $A^n \times X \longrightarrow B$, so we simply take f_n. For $m \geq 1$, $(\delta(f)_n)_m$ has type

$$(A^n \times A)^m \times (A^n \times X) \longrightarrow B.$$

Given an element

$$Z \xrightarrow{z} (A^n x A)^m \times (A^n \times X)$$

of the domain, one can view it as a matrix of elements

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} & a_{1*} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{r1} & \ldots & a_{rn} & a_{r*} \\ a_{*1} & \ldots & a_{*n} & x \end{pmatrix}$$

We then define $z(\delta(f)_n)_m$ by the formula

$$\sum_{s \leq n \& s \leq r} \langle a_{\alpha_1 1}, \ldots a_{\alpha_n n}, a_{\gamma_1 *} \ldots a_{\gamma_{r-s} *}, x \rangle f_{m+n-s}$$

where α and γ are defined by considering all possible ways of choosing s arguments of the form a_{ij}, $n - s$ arguments of the form a_{*j}, and $r - s$ arguments of the form a_{j*} so that the selected arguments include just one from each column and one from each row in the above matrix. For example, $(\delta(f)_1)_2$ takes an element

$$\begin{pmatrix} a_{11} & a_{1*} \\ a_{21} & a_{2*} \\ a_{*1} & x \end{pmatrix}$$

and returns

$$\langle a_{*1}, a_{1*}, a_{2*}, x\rangle f_3 + \langle a_{21}, a_{1*}, x\rangle f_2 + \langle a_{11}, a_{2*}, x\rangle f_2.$$

For more examples and discussion of this definition, see pages 401–405 of (Cockett and Seely 2011). Fortunately, as before, the hard work of proving this is a comonad has been done. (The corresponding restriction version of this construction, however, will require more work).

Theorem 2.11. **Faà** has the structure of a comonad, with counit $\epsilon : \mathbf{Fa\grave{a}}(\mathbb{X}) \longrightarrow \mathbb{X}$ given by:

— $\epsilon((A, +, 0), X) = X$,
— $\epsilon(f_*, f_1, f_2, \ldots) = f_*$,

and comultiplication $\delta : \mathbf{Fa\grave{a}}(\mathbb{X}) \longrightarrow \mathbf{Fa\grave{a}}^2(\mathbb{X})$ given by:

— $\delta((A, +, 0), X) = ((((A, +, 0), A), (+, \pi_0+, 0, \ldots), (0, \pi_0 0, 0, \ldots)), ((A, +, 0), X))$,
— action on arrows as above.

Proof. Again, the hard work has been done in theorem 2.2.2 of (Cockett and Seely 2011). The only thing extra needed to check here is that

$$((((A, +, 0), A), (+, \pi_0+, 0, \ldots), (0, \pi_0 0, 0, \ldots)), ((A, +, 0), X))$$

is an object of $\mathbf{Fa\grave{a}}^2(\mathbb{X})$, and this was done in the previous corollary. □

We now obtain our improved version of theorem 3.2.4 of (Cockett and Seely 2011).

Theorem 2.12. The coalgebras for the comonad **Faà** are exactly the generalized Cartesian differential categories.

Proof. If we have a coalgebra $\mathcal{D} : \mathbb{X} \longrightarrow \mathbf{Fa\grave{a}}(\mathbb{X})$, we let $\mathcal{D}(X) = (\mathcal{D}_0(X), \mathcal{D}_1(X))$. Since $\mathcal{D}$ satisfies the counit equations, we must have $\mathcal{D}_1(X) = X$. We define $L(X) := \mathcal{D}_0(X)$, and $D(f) := [\mathcal{D}(f)]_1$. Since $\mathcal{D}$ preserves products, we have $L(X \times Y) = L(X) \times L(Y)$.

Writing $L(X)$ as $(L_0(X), +_X, 0_X)$, the coassociativity equation

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\mathcal{D}} & \mathbf{Fa\grave{a}}(\mathbb{X}) \\ {\scriptstyle\mathcal{D}}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathbf{Fa\grave{a}}(\mathbb{X}) & \xrightarrow[\mathbf{Fa\grave{a}}(\mathcal{D})]{} & \mathbf{Fa\grave{a}}^2(\mathbb{X}) \end{array}$$

on objects tells us that

$$((L(L_0(X), L_0(X)), \mathcal{D}(+_X), \mathcal{D}(e_X), (L(X), X))$$

$$= (((L(X), L_0(X)), (+_X, \pi_0+_X, 0, \ldots), (0_X, \pi_0 0_X, 0, \ldots)), (L(X), X)),$$

so that we get

$$L(L_0(X)) = L(X), D(+_X) = \pi_0+_X, \text{ and } D(0_X) = \pi_0 0_X,$$

that is, we have **[CD.1]**. The equations **[CD.2]**–**[CD.7]** follow exactly as in theorem 3.2.4 of (Cockett and Seely 2011).

Conversely, if we have a generalized Cartesian differential category, we define

$$\mathcal{D}(X) := (L(X), X)$$

and

$$\mathcal{D}(f) := (f, D(f), D_2(f), D_3(f), \ldots)$$

where

$$D_n(f) := \langle 0, 0, \ldots 0, \pi_0, \pi_1, \ldots \pi_n \rangle D^n(f).$$

Almost all of the work in showing that this is a coalgebra is in theorem 3.2.4 of (Cockett and Seely 2011). The only thing left to check is that $\mathcal{D}(+) = (+, \pi_0+, 0, \ldots)$ and $\mathcal{D}(0) = (0, \pi_0 0, 0, \ldots)$. But **[CD.1]** gives $\mathcal{D}(+)_1 = \pi_0+$, and the higher terms are then 0, as

$$\langle 0, \pi_0, \pi_1, \pi_2 \rangle D^2(+) = \langle 0, \pi_0, \pi_1, \pi_2 \rangle D(\pi_0+) = \langle 0, \pi_0, \pi_1, \pi_2 \rangle \pi_0 \pi_0 D(+) = 0$$

and similarly for $\mathcal{D}(0)$. □

Corollary 2.13. If $\mathbb{X}$ is a Cartesian category, then **Faà**($\mathbb{X}$) is a generalized Cartesian differential category, with

$$D(f) = [\delta(f)]_1.$$

Of course, this is nothing more than stating that cofree coalgebras exist, but it is worth highlighting this particular result, as it shows that there are innumerable examples of generalized Cartesian differential categories. Note that **Faà**($\mathbb{X}$) has trivial differential structure if and only if 1 is the only commutative monoid in $\mathbb{X}$, as for example happens if $\mathbb{X}$ is a poset with finite meets. But in most cases of interest (say, $\mathbb{X}$ = **sets**), **Faà**($\mathbb{X}$) is highly non-trivial.

A more in-depth investigation of such cofree generalized Cartesian differential categories is clearly required; for now, we content ourselves with determining their linear maps (this was not considered in (Cockett and Seely 2011)).

In a Cartesian differential category, a map $f : X \longrightarrow Y$ is called linear if $D(f) = \pi_0 f$. For a general map in a generalized Cartesian differential category, this is not possible, as $\pi_0 f$ is not even well-defined. But if $L_0(X) = X$ and $L_0(Y) = Y$, then the types do match, and we can define what it means for such maps to be linear.

Definition 2.14. Say that an object X in a generalized Cartesian differential category is a **linear object** if $L_0(X) = X$. Say that map $f : X \longrightarrow Y$ between linear objects is a **linear map** if $D(f) = \pi_0 f$.

Recalling the definition of λ from proposition 2.9, we can now determine the linear maps in cofree generalized Cartesian differential categories.

Proposition 2.15. The linear maps in **Faà**($\mathbb{X}$) are exactly those maps of the form $\lambda(f)$ for f an additive map from $(A, +_A, 0_A)$ to $(B, +_B, 0_B)$.

Proof. Note that the linear objects in **Faà**($\mathbb{X}$) are those objects of the form

$$\lambda(A, +_A, 0_A) = ((A, +_A, 0_A), A).$$

Now suppose

$$((A,+_A,0_A,A)) \xrightarrow{(f_*,f_1,f_2,\ldots)} ((B,+_B,0_B),B)$$

is a linear map. We want to show that $f = \lambda(f_*)$, ie., that

$$f_1 = \pi_0 f_* \text{ and } f_n = 0 \ \forall n \geq 2.$$

Since f is linear, we have

$$(\pi_0 f)_* = (D(f))_* = (\delta(f)_1)_*.$$

But $(\pi_0 f)_* = \pi_0 f_*$ and from the definition of δ, $(\delta(f)_1)_* = f_1$, so we have $f_1 = \pi_0 f_*$.

We shall now prove that for all $n \geq 2$, $f_n = 0$ by induction on n. For the case $n = 2$, we know that $(\pi_0 f)_1 = (\delta(f)_1)_1$, so by the definition of δ and composition in $\mathbf{Fa\grave{a}}(X)$, we have

$$\langle \pi_0\pi_0, \pi_1\pi_0 \rangle f_1 = \langle \pi_0\pi_1, \pi_1\pi_0, \pi_1\pi_1 \rangle f_2 + \langle \pi_0\pi_0, \pi_1\pi_1 \rangle f_1.$$

In particular, in a context $\langle a, b, c, x \rangle$, we have

$$\langle a, c \rangle f_1 = \langle b, c, x \rangle f_2 + \langle a, x \rangle f_1.$$

Then setting $a = 0$ and recalling that f_1 is linear in its first variable, we have

$$0 = \langle b, c, x \rangle f_2$$

For any b, c, x. Thus $f_2 = 0$.

For $n > 2$, we also have $(\pi_0 f)_{n-1} = (\delta(f)_1)_{n-1}$. For the left side, by the definition of composition in $\mathbf{Fa\grave{a}}(\mathbb{X})$, $(\pi_0 f)_{n-1}$ is a sum over certain binary trees τ. But with the exception of the binary tree with a single node out of the root, the expression $(\pi_0 \star f)_{n-1}$ will have f_i applied to at least one term with 0; since each f_i is additive, these expression are then 0. For the tree with a single node out of the root, we have f_{n-1} applied to some terms. But by the induction assumption, f_{n-1} is 0, so we have $(\pi_0 f)_{n-1} = 0$.

For the right side, recalling the definition of $(\delta(f)_1)_{n-1}$ from (Cockett and Seely 2011), the only possible choices for the index s are 0 or 1 (as in this case $r = 1$). $(\delta(f)_1)_{n-1}$ is then a sum of terms, one of which is f_n, the other terms f_{n-1} applied at some value. However, by the induction assumption, $f_{n-1} = 0$, so $(\delta(f)_1)_{n-1} = f_n$. Putting this together with the above gives $f_n = 0$, as required.

We have thus shown that if f is linear, then f must be of the form $\lambda(f_*)$; conversely, the above calculations also show that maps of such form are linear. □

3. Differential restriction categories revisited

As the first part of this paper generalized the Cartesian differential categories of (Blute et. al. 2009), so this second part generalizes the differential restriction categories of (Cockett et. al 2011). The immediate benefit of the generalized version is that, unlike with ordinary differential restriction categories, splitting the restriction idempotents of a differential restriction category retains differential structure. The other important aspect we shall describe is the restriction version of the Faà di bruno comonad. (Note that this has not been explored in even the non-generalized version). This will require much more care than the non-restriction generalization given in the previous section, as we have to ensure everything works well with the restriction structure.

As noted in the introduction, since every category is a restriction category with trivial restriction structure, the results of the previous section are in fact corollaries of the results presented in this section. However, we have presented the basic non-restriction version first to help the reader gain an understanding of the constructions and definitions involved before seeing how the structure interacts with restrictions.

To begin, we first recall the definition of a restriction category from (Cockett and Lack 2002):

Definition 3.1. Given a category, $\mathbb{X}$, a **restriction structure** on $\mathbb{X}$ gives for each map $A \xrightarrow{f} B$, a map, $A \xrightarrow{\overline{f}} A$, satisfying four axioms:

[**R.1**] $\overline{f}\, f = f$;
[**R.2**] If $dom(f) = dom(g)$ then $\overline{g}\,\overline{f} = \overline{f}\,\overline{g}$;
[**R.3**] If $dom(f) = dom(g)$ then $\overline{\overline{g}\, f} = \overline{g}\,\overline{f}$;
[**R.4**] If $dom(h) = cod(f)$ then $f\overline{h} = \overline{fh}\, f$.

A category with a specified restriction structure is a **restriction category**.

The canonical example is that of partial functions between sets, where, given a partial function $f : X \longrightarrow Y$, $\overline{f}$ is essentially the identity on the domain of definition of f:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Recall also that map in a restriction category is said to be **total** if $\overline{f} = 1$, while a **restriction idempotent** is a map $e : X \longrightarrow X$ such that $\overline{e} = e$.

The following results from (Cockett and Lack 2002) are useful when doing calculations with the restriction operation:

Proposition 3.2. If $\mathbb{X}$ is a restriction category then:

(i) $\overline{f}$ is idempotent;
(ii) $\overline{f}\,\overline{fg} = \overline{fg}$;
(iii) $\overline{f\overline{g}} = \overline{fg}$;
(iv) $\overline{\overline{f}} = \overline{f}$;
(v) $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}$;
(vi) If f is monic then $\overline{f} = 1$ (and so in particular $\overline{1} = 1$);
(vii) $\overline{f}\, g = g$ implies $\overline{g} = \overline{f}\,\overline{g}$.

In addition, we need to recall what it means for a restriction category to have finite products. Thinking about the category of sets and partial functions, one sees that the ordinary product of sets X and Y will not a be a categorical product in the category of sets and partial functions. Given maps $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$, we will not have $\langle f, g\rangle\pi_0 = f$, as $\langle f, g\rangle$ will only be defined where both f and g are. However, $\langle f, g\rangle\pi_0$ will be equal to f on this smaller domain of definition. This leads to the following definitions in an abstract restriction category.

Definition 3.3. Given parallel maps $f, g : A \longrightarrow B$ in a restriction category, write $f \leq g$ if $\overline{f}\, g = f$, and write $f \smile g$ if $\overline{f}\, g = \overline{g}\, f$.

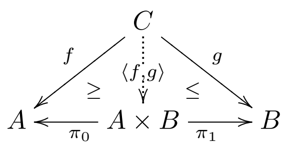
Intuitively, $f \leq g$ captures the idea of g being the same function as f, but on a larger domain of definition, while $f \smile g$ captures the idea of f and g being equal wherever they are both defined. One can check that $\leq$ is a partial order. We can now define what it means for a restriction category to have restriction products.

Definition 3.4. Let $\mathbb{X}$ be a restriction category. A **restriction terminal object** is an object T in $\mathbb{X}$ such that for any object A, there is a unique total map $!_A : A \longrightarrow T$ which satisfies $!_T = 1_T$. These maps must also satisfy the property that for any map $f : A \longrightarrow B$, $f!_B \leq !_A$, i.e. $f!_B = \overline{f!_B}\ !_A = \overline{f\overline{!_B}}\ !_A = \overline{f}\ !_A$.

A **restriction product** of objects A, B in $\mathbb{X}$ consists of an object $A \times B$ with two total maps

$$\pi_0 : A \times B \longrightarrow A \qquad \pi_1 : A \times B \longrightarrow B$$

satisfying the property that for any object C and maps $f : C \longrightarrow A, g : C \longrightarrow B$ there is a unique map $\langle f, g\rangle : C \longrightarrow A \times B$ such that both triangles below exhibit lax commutativity



that is,

$$\langle f, g\rangle\pi_0 = \overline{\langle f, g\rangle}f \quad \text{and} \quad \langle f, g\rangle\pi_1 = \overline{\langle f, g\rangle}g.$$

In addition, we ask that $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}$. A restriction category $\mathbb{X}$ is a **cartesian restriction category** if $\mathbb{X}$ has a restriction terminal object and all restriction products.

This was first described in (Cockett and Lack 2007) where it was noted that the resulting structure was equivalent to the P-categories introduced in (Robinson and Rosolini 1988).

We now generalize the definition of a differential restriction category from (Cockett et. al 2011).

Definition 3.5. A **generalized differential restriction category** is a Cartesian restriction category with:

— for each object X, a total commutative monoid $L(X) = (L_0(X), +_X, 0_X)$, satisfying

$$L(L_0(X)) = L(X) \text{ and } L(X \times Y) = L(X) \times L(Y),$$

— for each map $f : X \longrightarrow Y$, a map $D(f) : L_0(X) \times X \longrightarrow L_0(Y)$ such that:

[DR.1] $D(+_X) = \pi_0 +_X$ and $D(0_X) = \pi_0 0_X$;

[DR.2] $\langle a + b, c\rangle D(f) = \langle a, c\rangle D(f) + \langle b, c\rangle D(f)$ and $\langle 0, a\rangle D(f) = \overline{af}0$;

[DR.3] $D(\pi_0) = \pi_0\pi_0$, and $D(\pi_1) = \pi_0\pi_1$;

[DR.4] $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$;

[DR.5] $D(fg) = \langle D(f), \pi_1 f\rangle D(g)$;
[DR.6] $\langle\langle a, 0\rangle, \langle c, d\rangle\rangle D^2(f) = \overline{c}\langle a, d\rangle D(f)$;
[DR.7] $\langle\langle 0, b\rangle, \langle c, d\rangle\rangle D^2(f) = \langle\langle 0, c\rangle, \langle b, d\rangle\rangle D^2(f)$;
[DR.8] $D(\overline{f}) = (1 \times \overline{f})\pi_0 = \overline{\pi_1 f}\,\pi_0$;
[DR.9] $\overline{D(f)} = 1 \times \overline{f} = \overline{\pi_1 f}$.

Note the addition of a restriction in axioms **[DR.2]** and **[DR.6]**: this is necessary since we must keep track of the partiality of the maps that are lost across the equality. The 8th and 9th axioms demand that the differential operator be total in its first variable.

Remark 3.6. Any generalized Cartesian differential category is a generalized differential restriction category, when equipped with the trivial restriction structure ($\overline{f} = 1$ for all f). Any differential restriction category is a generalized differential restriction category, with $L(X) = X$ for each X.

The standard example of a non-trivial differential restriction category is:

Example 3.7. Smooth functions defined on open subsets of $\mathcal{R}^n$.

More examples, such as differential restriction categories of rational functions, can be found in (Cockett et. al 2011).

One of the reasons for generalizing differential restriction categories is the following construction. Recall from (Cockett and Lack 2002) that if $\mathbb{X}$ is a restriction category, then the **restriction idempotent splitting of** $\mathbb{X}$, $K_r(\mathbb{X})$, is a restriction category with:
— objects restriction idempotents $(X, e = \overline{e})$;
— a map $f : (X, e_1) \longrightarrow (Y, e_2)$ is a map $f : X \longrightarrow Y$ such that $e_1 f e_2 = f$;
— restriction and composition as in $\mathbb{X}$, and $1_{(X,e)} := e$.

One problem with differential restriction categories is that even if $\mathbb{X}$ is a differential restriction category, $K_r(\mathbb{X})$ need not be: because of [**DR**.**9**], the derivative must be total in the first variable, and so the derivative of a map $f : (X, e_1) \longrightarrow (Y, e_2)$ cannot have domain $(X, e_1) \times (X, e_1)$. With generalized differential restriction categories, this is no longer a problem, as we can set $L(X, e_1) = (L(X), 1)$.

Proposition 3.8. If $\mathbb{X}$ is a generalized differential restriction category then $K_r(\mathbb{X})$ is also, with

$$L(X, e) := (L(X), 1) \text{ and } D(f) := D(f).$$

Proof. The only thing to check is that $D(f)$ is a valid map in the restriction idempotent splitting category. Suppose $f : (X, e_1) \longrightarrow (Y, e_2)$. We are claiming that $D(f)$ is a valid map from $(L_0(X) \times X, 1 \times e_1)$ to $(L_0(Y), 1)$. So consider

$$(1 \times e_1)D(f) = (1 \times e_1)\overline{D(f)}\,D(f) = (1 \times e_1)(1 \times \overline{f}\,)D(f) = (1 \times \overline{f}\,)D(f) = D(f)$$

by [**DR**.**9**] and the fact that $e_1 f e_2 = f$. □

For example, when $\mathbb{X}$ is the differential restriction category of partial smooth maps between Cartesian spaces, $K_r(\mathbb{X})$ is a generalized differential restriction category whose objects are open subsets of Cartesian spaces.

Corollary 3.9. If $\mathbb{X}$ is a differential restriction category, then the total maps of $K_r(\mathbb{X})$ form a generalized Cartesian differential category.

As noted in the introduction, this shows that the categories whose objects are open subsets of $\mathbb{R}^n$, and whose maps are (total) smooth maps between them, forms a generalized Cartesian differential category.

3.1. *Faà di bruno - restriction version*

In this section, we extend the Faàdi Bruno comonad to work with restriction categories. Recall from earlier that in $\mathbf{Fa\grave{a}}(\mathbb{X})$, given two composable maps $f : (A, X) \longrightarrow (B, Y)$, $g : (B, Y) \longrightarrow (C, Z)$, we have

$$(fg)_n := \sum (f \star g)_\tau,$$

where the sum is over each tree τ of length 2 and width n. For such a tree τ, the term $(f \star g)_\tau$ term is of the form

$$\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots p_k f_{i_k}, \pi_n f_* \rangle g_k$$

where k, each i_j, and p_i all depend on the tree τ, and each p_i is of the form $\langle \pi_{j_1}, \pi_{j_2}, \ldots \pi_{j_{i_1}}, \pi_n \rangle$.

Our first goal is to prove the following

Theorem 3.10. Given a Cartesian restriction category $\mathbb{X}$, there is a Cartesian restriction category $\mathbf{Fa\grave{a}}(\mathbb{X})$ which has:

— objects pairs $((A, +, 0), X)$, where X is an object of $\mathbb{X}$ and $(A, +, 0)$ is a (total) commutative monoid in $\mathbb{X}$;

— maps sequences

$$(f_*, f_1, f_2, \ldots) : (A, X) \longrightarrow (B, Y)$$

where $f_* : X \longrightarrow Y$, and for each $n \geq 1$, $f_n : (A)^n \times X \longrightarrow B$ with f_n additive and symmetric in the first n variables and

$$\overline{f_n} = 1 \times \overline{f_*} = \overline{\pi_n f_*};$$

— composition and identities are defined as in the total case;

— restriction given by

$$\overline{(f_*, f_1, f_2, \ldots)} := (\overline{f_*}, \overline{\pi_1 f_*}\, \pi_0, \overline{\pi_2 f_*}\, 0, \overline{\pi_3 f_*}\, 0 \ldots).$$

We will begin with a lemma. When working with a putative restriction category, it is often helpful to first determine what a map of the form $\overline{f}\, g$ looks like.

Lemma 3.11. With the above definition of $\overline{f}$, for $n \geq 1$ and maps $f : X \longrightarrow A$, $g : X \longrightarrow B$, we have

$$(\overline{f}\, g)_n = \overline{\pi_n f_*}\, g_n.$$

Proof. Recalling the definition of composition as above, we have $(\overline{f} \star g)_\tau$ is of the form

$$\langle p_1 \overline{f}_{i_1}, p_2 \overline{f}_{i_2}, \ldots p_k \overline{f}_{i_k}, \pi_n \overline{f}_* \rangle g_k$$

But for any $i \geq 1$, the expression $p_j \overline{f}_{i_j}$ is a restriction of 0, and with the exception of the tree with n branches out of the root, that expression occurs at least once. Then since each g_k is additive in each of the first n variables, we have

$$(\overline{f} \star g)_\tau = \overline{\pi_n \overline{f_*}\, g_*}\, 0 = \overline{\pi_n f_*}\, \overline{\pi_n g_*}\, 0.$$

For that one tree τ_0 with n branches out of the root,

$$(\overline{f} \star g)_{\tau_0} = \langle \langle \pi_0, \pi_n \rangle (\overline{f}\,)_1, \langle \pi_1, \pi_n \rangle (\overline{f}\,)_1, \ldots \langle \pi_{n-1}, \pi_n \rangle (\overline{f}\,)_1, \pi_n f_* \rangle g_n.$$

But for each $i \in \{1 \ldots n-1\}$,

$$\langle \pi_i, \pi_n \rangle (\overline{f}\,)_1 = \langle \pi_i, \pi_n \rangle \overline{\pi_1 f_*}\, \pi_0 = \overline{\pi_n f_*}\, \langle \pi_i, \pi_n \rangle \pi_0 = \overline{\pi_n f_*}\, \pi_i,$$

so that

$$(f \star g)_{\tau_0} = \overline{\pi_n f_*}\, \langle \pi_0, \pi_1, \ldots \pi_n \rangle g_n = \overline{\pi_n f_*}\, g_n$$

Thus, the sum over all trees τ equals

$$\overline{\pi_n f_*}\, g_n + \overline{\pi_n f_*}\, \overline{\pi_n g_*}\, 0 = \overline{\pi_n f_*}\, \overline{\pi_n g_*}\, g_n = \overline{\pi_n f_*}\, g_n$$

by assumption on g_n. Thus we have

$$(\overline{f}\, g)_n = \overline{\pi_n f_*}\, g_n,$$

as required. □

Proposition 3.12. $\mathbf{Fa\grave{a}}(\mathbb{X})$ is a restriction category.

Proof.

We first need to check that the identities, composites, and restriction satisfy the added requirement on the restriction of its components. That identities satisfy the requirement is obvious, since $1_{(A,x)} = (1, \pi_0, 0_A, \ldots)$ and both π_0 and 0_A are total.

To check that the composite of two maps $f : (A, X) \longrightarrow (B, Y)$, $g : (B, Y) \longrightarrow (C, Z)$ satisfies the restriction requirement, as above, recall that $\overline{(fg)_n} = \sum (f \star g)_\tau$. Since $\overline{x+y} = \overline{x}\,\overline{y}$, to calculate $\overline{(fg)_n}$, we need to calculate each $\overline{(f \star g)_\tau}$. For such a tree τ, this term is of the form

$$\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots p_k f_{i_k}, \pi_n f_* \rangle g_k$$

where k, each i_j, and p_i all depend on the tree τ. In particular, however, each p_m is of the form $\langle \pi_{j_1}, \pi_{j_2}, \ldots \pi_{j_{i_m}}, \pi_n \rangle$ (where again each j_l depends on τ) so we have

$$\begin{aligned}
& \overline{p_m f_{i_m}} \\
= \; & \overline{\langle \pi_{j_1}, \pi_{j_2}, \ldots \pi_{j_{i_m}}, \pi_n \rangle f_{i_m}} \\
= \; & \overline{\langle \pi_{j_1}, \pi_{j_2}, \ldots \pi_{j_{i_m}}, \pi_n \rangle \overline{f_{i_m}}} \\
= \; & \overline{\langle \pi_{j_1}, \pi_{j_2}, \ldots \pi_{j_{i_m}}, \pi_n \rangle \overline{\pi_{i_m} f_*}} \\
= \; & \overline{\langle \pi_{j_1}, \pi_{j_2}, \ldots \pi_{j_{i_m}}, \pi_n \rangle \pi_{i_m} f_*} \\
= \; & \overline{\pi_n f_*}
\end{aligned}$$

Then we can calculate, for any such tree τ,

$$\begin{aligned}
\overline{(f \star g)_\tau} &= \overline{\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots p_k f_{i_k}, \pi_n f_* \rangle g_k} \\
&= \overline{\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots p_k f_{i_k}, \pi_n f_* \rangle \overline{g_k}} \\
&= \overline{\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots p_k f_{i_k}, \pi_n f_* \rangle \overline{\pi_n g_*}} \\
&= \overline{\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots p_k f_{i_k}, \pi_n f_* \rangle \pi_n g_*} \\
&= \overline{\overline{p_1 f_{i_1}}\,\overline{p_2 f_{i_2}}\ \ldots \overline{p_k f_{i_k}}\ \pi_n f_* g_*} \\
&= \overline{\overline{\pi_n f_*}\,\overline{\pi_n f_*}\ \ldots \overline{\pi_n f_*}\ \pi_n f_* g_*} \\
&= \overline{\pi_n f_* g_*} \\
&= \overline{\pi_n (fg)_*}
\end{aligned}$$

so that we get $\overline{(fg)_n} = \overline{\pi_n (fg)_*}$, as required.

Each restriction map satisfies the requirement on the restriction of its components since

$$\overline{\overline{\pi_n f_*}\,\pi_0} = \overline{\pi_n}\, f_* = \overline{\overline{\pi_n f_*}\,0}$$

as 0 and π_0 are both total.

That **Faà**($\mathbb{X}$) is a category is as in (Cockett and Seely 2011). We now turn to checking the restriction axioms. As the equality of $*$ terms follows directly, we will simply check for the $n \geq 1$ terms. For [**R.1**], by lemma 3.11,

$$(\overline{f}\, f)_n = \overline{\pi_n f_*}\, f_n = f_n$$

by assumption on f_n. For [**R.2**], for $n \geq 2$, by lemma 3.11, we have

$$(\overline{f}\,\overline{g}\,)_n = \overline{\pi_n f_*}\,\overline{\pi_n g_*}\,0 = \overline{\pi_n g_*}\,\overline{\pi_n f_*}\,0 = (\overline{g}\,\overline{f}\,)_n$$

and $n = 1$ is similar. For [**R.3**], again by lemma 3.11, we have

$$(\overline{\overline{f}\, g}\,)_1 = \overline{\pi_1 (\overline{f}\, g)_*}\,0 = \overline{\pi_1 \overline{f_*}\, g}\,0 = \overline{\overline{\pi_1 f_*}\,\pi_1 g}\,0 = \overline{\pi_1 f_*}\,\overline{\pi_1 g_*}\,0$$

which is the required value, by the calcuation for [**R.2**]; again $n = 1$ is similar.

For [**R.4**], we need to find the nth term of $f\overline{g}$. This time, for any tree τ with the exception of the tree with a *single* node out of the root, we have

$$\begin{aligned}
(f \star \overline{g}\,)_\tau &= \langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots \pi_n f_* \rangle \overline{\pi_n g_*}\,0 \\
&= \overline{\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots \pi_n f_* \rangle \pi_n g_*}\,\langle p_1 f_{i_1}, p_2 f_{i_2}, \ldots \pi_n f_* \rangle\ 0 \\
&= \overline{\pi_n f_* g_*}\,\overline{\pi_n f_*}\,0 \\
&= \overline{\pi_n f_* g_*}\,0
\end{aligned}$$

For that one tree τ_1 with a single node out of the root, we have

$$(f \star \overline{g}\,)_{\tau_1} = \langle f_n, \pi_n f_* \rangle \overline{\pi_1 g_*}\,\pi_0 = \overline{\pi_n f_* g_n}\, f_n.$$

Then summing over all trees τ gives

$$(f\overline{g}\,)_n = \overline{\pi_n f_* g_*}\,0 + \overline{\pi_n f_* g_n}\, f_n = \overline{\pi_n f_* g_n}\, f_n.$$

Conversely, by lemma 3.11, we have

$$(\overline{fg}\, f)_n = \overline{\pi_n (fg)_*}\, f_n = \overline{\pi_n f_* g_*}\, f_n$$

so that [**R.4**] is satisfied, as required.

□

To prove that **Faà**($\mathbb{X}$) is a *Cartesian* restriction category, we need to determine which maps are total, and when $\leq$ and $\smile$ hold.

Proposition 3.13. In the restriction category **Faà**($\mathbb{X}$), for maps $f, g : (X, A) \longrightarrow (Y, B)$:

(i) f is total if and only if f_* is total;
(ii) $f \leq g$ if and only if $f_* \leq g_*$ and $f_n \leq g_n$ for each $n \geq 1$;
(iii) $f \smile g$ if and only if $f_* \smile g_*$ and $f_n \smile g_n$ for each $n \geq 1$.

Proof.

(i)If f is total, then in particular $(\overline{f}\,)_* = 1$, so f_* is total. Conversely, if f_* is total, then for $n \geq 2$,

$$(\overline{f}\,)_n = \overline{\pi_n f_*}\, 0 = \overline{\pi_n}\, 0 = 0$$

and similarly $(\overline{f}\,)_1 = \pi_0$, so that f is total.

(ii)Recall that $f \leq g$ means $\overline{f}\, g = f$. So $f \leq g$ if and only if $f_* \leq g_*$ and for each $n \geq 1$, $(\overline{f}\, g)_n = f_n$. But by lemma 3.11,

$$(\overline{f}\, g)_n = \overline{\pi_n f_*}\, g_n = \overline{f_n}\, g_n$$

by assumption on f_n. Thus $f \leq g$ if and only if $f_* \leq g_*$ and for each $n \geq 1$, $f_n \leq g_n$.

(iii)Recall that $f \smile g$ means $\overline{f}\, g = \overline{g}\, f$. The result then follows as in (ii).

□

We can now prove the following.

Proposition 3.14. With product structure as in the total case:

$$\pi_i = (\pi_i, \pi_0 \pi_i, 0, 0, \ldots), \langle f, g \rangle_n := \langle f_n, g_n \rangle,$$

Faà($\mathbb{X}$) is a Cartesian restriction category.

Proof. We first need to show $\langle f, g \rangle \pi_0 \leq f$, so consider the term $(\langle f, g \rangle \pi_0)_n$. As in the proof of [**R.4**] in Theorem 3.10, for any tree τ with the exception of the tree τ_1 which has a single node coming out of the root, $(\langle f, g \rangle \star \pi_0)_\tau$ is of the form

$$\begin{aligned}
& \langle p_1 \langle f, g \rangle_{i_1}, p_2 \langle f, g \rangle_{i_2}, \ldots \pi_n \langle f, g \rangle_* \rangle 0 \\
= \;& \overline{p_1 \langle f, g \rangle_{i_1}}\, \overline{p_2 \langle f, g \rangle_{i_2}} \ldots \overline{\pi_n \langle f, g \rangle_*}\, 0 \\
= \;& \overline{\pi_n \langle f_*, g_* \rangle}\, 0 \\
= \;& \overline{\pi_n f_*}\, \overline{\pi_n g_*}\, 0
\end{aligned}$$

while for τ_1,

$$\begin{array}{rcl}(\langle f,g\rangle \star \pi_0)_{\tau_1} & = & \langle(\langle f,g\rangle_n, \pi_n\langle f,g\rangle_*\rangle\pi_0\pi_0 \\ & = & \langle\langle f_n,g_n\rangle, \pi_n\langle f_*,g_*\rangle\rangle\pi_0\pi_0 \\ & = & \overline{g_n}\,\overline{\pi_n f_*}\,\overline{\pi_n g_*}\,f_n \\ & = & \overline{g_n}\,f_n \text{ (by assumption on } g_n)\end{array}$$

Thus

$$(\langle f,g\rangle\pi_0)_n = \overline{\pi_n f_*}\,\overline{\pi_n g_*}\,0 + \overline{g_n}\,f_n = \overline{g_n}\,f_n,$$

so that by lemma 3.13, $\langle f,g\rangle\pi_0 \leq f$, as required; π_1 is similar.

We also need to show that $\overline{\langle f,g\rangle} = \overline{f}\,\overline{g}$. For $n \geq 2$, consider

$$(\overline{\langle f,g\rangle}\,)_n = \overline{\pi_n\langle f_g\rangle_*}\,0 = \overline{\pi_n\langle f_*,g_*\rangle}\,0 = \overline{\pi_n f_*}\,\overline{\pi_n g_*}\,0$$

while by lemma 3.11,

$$(\overline{f}\,\overline{g}\,)_n = \overline{\pi_n f_*}\,(\overline{g}\,)_n = \overline{\pi_n f_*}\,\overline{\pi_n g_*}\,0,$$

and $n = 1$ is similar, so that $\overline{\langle f,g\rangle} = \overline{f}\,\overline{g}$, as required. □

This completes the proof of theorem 3.10. We now turn to the monad structure of **Faà**.

Proposition 3.15. **Faà** extends to an endofunctor on the category of Cartesian restriction categories (where the maps are those functors which preserve products and restrictions exactly), where we define

$$\mathbf{Fa\grave{a}}(F)(f_*, f_1, f_2, \ldots) := (F(f_*), F(f_1), F(f_2), \ldots)$$

Proof. The only thing to check is that **Faà**(F) satisfies the restriction requirement on its components:

$$\overline{F(f_n)} = F(\overline{f_n}\,) = F(\overline{\pi_n f_*}\,) = \overline{\pi_n F(f_*)}$$

since F preserves restrictions and products exactly. □

Proposition 3.16. The endofunctor **Faà** has the structure of a comonad, with counit $\epsilon : \mathbf{Fa\grave{a}}(\mathbb{X}) \longrightarrow \mathbb{X}$ given by:

— $\epsilon((A,+,0),X) = X$,
— $\epsilon(f_*, f_1, f_2, \ldots) = f_*$,

and comultiplication $\delta : \mathbf{Fa\grave{a}}(\mathbb{X}) \longrightarrow \mathbf{Fa\grave{a}}^2(\mathbb{X})$ given by:

— $\delta((A,+,0),X) = (((A,+,0),A),(+,\pi_0+,0,\ldots),(0,\pi_0 0,0,\ldots)),((A,+,0),X)))$,
— action on arrows as in the total case.

Proof. There are only a few additional things to check here:

(i) that ϵ preserves restrictions;
(ii) that $\delta(f)$ is a valid arrow in $\mathbf{Fa\grave{a}}^2(\mathbb{X})$;
(iii) that δ preserves restrictions.

The first part is obvious, as by definition $(\overline{f}\,)_* = \overline{f_*}$.

For (ii), we need to show that

$$\overline{\delta(f)_n} = \overline{\pi_n \delta(f)_*}\,,$$

so we need to show that they are equal in each component. For $m \geq 2$, we have

$$\begin{array}{rl} & (\overline{\delta(f)_n})_m \\ = & \overline{\pi_m (\delta(f)_n)_*}\,0 \text{ (by definition of restriction)} \\ = & \overline{\pi_m f_n}\,0 \text{ (by definition of } \delta(f)) \\ = & \overline{\pi_m \overline{f_n}}\,0 \\ = & \overline{\pi_m \overline{\pi_n f_*}}\,0 \text{ (by assumption on } f_n) \\ = & \overline{\pi_m \pi_n f_*}\,0 \end{array}$$

while

$$(\overline{\pi_n \delta(f)_*})_m = \overline{\pi_m (\pi_n \delta(f)_*)_*}\,0 = \overline{\pi_m \pi_n f_*}\,0$$

by definition of $\delta(f)$, so that they are equal, as required. The case $m = 1$ and the $*$ component are similar.

For (iii), we need $\delta(\overline{f}\,) = \overline{\delta(f)}\,$, so in particular we need for each $n, m \geq 1$,

$$((\delta(\overline{f}\,))_n)_m = ((\overline{\delta(f)}\,)_n)_m.$$

For $n, m \geq 2$, starting with the right side, we have

$$\begin{array}{rl} & ((\overline{\delta(f)}\,)_n)_m \\ = & (\overline{\pi_n \delta(f)_*}\,0)_m \text{ (by definition of restriction)} \\ = & \overline{\pi_m (\pi_n \delta(f)_*)_*}\,0 \text{ (by lemma 3.11)} \\ = & \overline{\pi_m \pi_n f_*}\,0 \text{ (by definition of } \delta). \end{array}$$

For $((\delta(\overline{f}\,))_n)_m$, recalling the definition of δ the previous section, we see that $((\delta(\overline{f}\,))_n)_m$ is a sum of terms of the form

$$\langle \pi_{\alpha_1,\beta_1}, \pi_{\alpha_2,\beta_2}, \ldots, \pi_m \rangle \overline{\pi_n f_*}\,0,$$

for certain indices α_i, β_j. The particular form of these indices is not important however, as in each case we have

$$\begin{array}{rl} & \langle \pi_{\alpha_1,\beta_1}, \pi_{\alpha_2,\beta_2}, \ldots, \pi_m \rangle \overline{\pi_n f_*}\,0 \\ = & \overline{\langle \pi_{\alpha_1,\beta_1}, \pi_{\alpha_2,\beta_2}, \ldots, \pi_m \rangle \pi_n f_*}\,\langle \pi_{\alpha_1,\beta_1}, \pi_{\alpha_2,\beta_2}, \ldots, \pi_m \rangle 0 \\ = & \overline{\pi_m \pi_n f_*}\,0 \text{ (since projections are total)} \end{array}$$

so that the sum is also $\overline{\pi_m \pi_n f_*}\,0$, and we have $((\delta(\overline{f}\,))_n)_m = ((\overline{\delta(f)}\,)_n)_m$. The cases for m, n equalling 1 or $*$ or similar, so $\delta(\overline{f}\,) = \overline{\delta(f)}\,$, as required. □

Theorem 3.17. The coalgebras for the comonad $(\mathbf{Fa\grave{a}}, \epsilon, \delta)$ are exactly the generalized differential restriction categories.

Proof. As before, if we have a coalgebra $\mathcal{D} : \mathbb{X} \longrightarrow \mathbf{Fa\grave{a}}(\mathbb{X})$, we let $\mathcal{D}(X) = (\mathcal{D}_0(X), \mathcal{D}_1(X))$.

Since $\mathcal{D}$ satisfies the counit equations, we must have $\mathcal{D}_1(X) = X$. We define $L(X) := \mathcal{D}_0(X)$, and $D(f) := [\mathcal{D}(f)]_1$.

For the most part, the fact that this operation satisfies the differential restriction axioms is exactly as before, with a few minor modifications. Since $D(f) := [\mathcal{D}(f)]_1$ is a map in **Faà**($\mathbb{X}$), we must have $\overline{D(f)_1} = \overline{\pi_1 \mathcal{D}(f)_*} = \overline{\pi_1 f}$. Since $\mathcal{D}$ preserves restrictions, we have

$$D(\overline{f}\,) = \mathcal{D}(\overline{f}\,)_1 = (\overline{\mathcal{D}(f)}\,)_1 = \overline{\pi_1 f}\,\pi_0.$$

Thus, we have both of the added differential restriction axioms.

Note that $D(f)$ being additive in its first variable means that

$$\langle 0, x\rangle D(f) = \overline{\langle 0, x\rangle [\mathcal{D}(f)]_1}\,0 = \overline{xf_*}\,0 = \overline{xf}\,0,$$

giving [**DR**.**2**]. Similarly, as on pg. 414 of (Cockett and Seely 2011), we get [**DR**.**6**] by setting a certain term equal to 0; the extra restriction term then comes out when we project.

Conversely, if we have a generalized Cartesian differential category, and define

$$\mathcal{D}(f) := (f, D(f), D_2(f), D_3(f), \ldots)$$

where

$$D_n(f) := \langle 0, 0, \ldots 0, \pi_0, \pi_1, \ldots \pi_n\rangle D^n(f),$$

the only thing we need to check here is that $\mathcal{D}(f)$ is a valid map in **Faà**($\mathbb{X}$). That is, we need $\overline{D_n(f)} = \overline{\pi_n f}$. But since $\overline{D(f)} = \overline{\pi_1 f}$, we have $\overline{D^n(f)} = \overline{\pi_1 \pi_1 \ldots \pi_1 f}$ (n π_1's), so that

$$\overline{D_n(f)} = \overline{\langle 0, 0, \ldots 0, \pi_0, \pi_1, \ldots \pi_n\rangle D^n(f)} = \overline{\pi_n f}\,,$$

as required.

□

Corollary 3.18. If $\mathbb{X}$ is a Cartesian restriction category, then **Faà**($\mathbb{X}$) is a generalized differential restriction category, with

$$D(f) := [\delta(f)]_1.$$

Again, this is nothing more than stating that free coalgebras exist; but it highlights the fact that there are many instances of generalized differential restriction categories beyond the standard examples.

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