Embeddings of atlas categories

Geoff Cruttwell
(joint work with Robin Cockett)

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Manifolds as Atlases or as Spaces?

Let us begin by considering a generalized version of topological manifolds:

Definition

A topological space X is a **real manifold** if X has a covering by open sets U_i, such that each U_i is homeomorphic to an open subset of some $\mathcal{R}^n$.

This seems nice...where do atlases come in?

Smooth Manifolds as Atlases

One problem is when you try to define "smooth" manifold. You would like to say the following:

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Except it doesn't make sense (an arbitrary topological space has no notion of smoothness)! So, instead, you have to consider "charts" (open subsets of $\mathcal{R}^n$) together with their transition functions, and ask that each transition function be smooth.

Smooth Manifolds as Spaces

Eventually, some clever people found ways around this:

- an open subset of $\mathcal{R}^n$ can be represented by the sheaf of smooth functions (an $\mathcal{R}$-algebra) on it, so we can define a smooth manifold as a sheaf of $\mathcal{R}$-algebras which locally looks like the sheaf of smooth functions on an open subset of $\mathcal{R}^n$;

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- "diffeological spaces" (we will review these later) have an inherent notion of smoothness, and every open subset of $\mathcal{R}^n$ is naturally a diffeological space, so we can define a smooth manifold to be a diffeological space which locally looks like an open subset of $\mathcal{R}^n$.

But can we do this type of thing for every notion of atlas?

Atlases again

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For example:

- a recent notion of Kriegl and Michor is that of a "convenient vector space": a locally convex space with a nice notion of smooth map between them;
- one can then look at atlases of convenient vector spaces, giving a definition of smooth manifolds modelled on infinite dimensional vector spaces;
- but the algebra of smooth functions to $\mathcal{R}$ need not determine the convenient vector space, so the "manifold as a sheaf of algebras" approach will not work!

Idea of the talk

This is one specific example: what if another definition of "manifold as an atlas of [some modelling spaces]" is defined? Can we be sure that such atlases always look like spaces?

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This is one specific example: what if another definition of "manifold as an atlas of [some modelling spaces]" is defined? Can we be sure that such atlases always look like spaces? To answer the question, we'll look at two things:

- What does it mean to say that an object is an altas of other objects? (Grandis' construction)
- Can every category of atlases be realized as a category of "spaces" which locally look like the modelling spaces?

I'll show that two possibilities exist for Grandis' general notion of "atlas".

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Definition

Given a category, $\mathbb{X}$, a **restriction structure** on $\mathbb{X}$ gives for each, $A \xrightarrow{f} B$, a restriction arrow, $A \xrightarrow{\overline{f}} A$, that satisfies four axioms:

[R.1] $\overline{f}\, f = f$;

[R.2] If $dom(f) = dom(g)$ then $\overline{g}\,\overline{f} = \overline{f}\,\overline{g}$;

[R.3] If $dom(f) = dom(g)$ then $\overline{\overline{g}\, f} = \overline{g}\,\overline{f}$;

[R.4] If $dom(g) = cod(f)$ then $f\overline{g} = \overline{fg}\, f$.

A category with a specified restriction structure is a **restriction category**.

Examples

Some examples:

- **set**: sets and partial functions, with

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

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- **fdSmooth**: objects $\mathcal{N}$, arrow $f : n \longrightarrow m$ is a smooth function $f : U \subseteq \mathcal{R}^n \longrightarrow \mathcal{R}^m$ (U open).

Aspects of Restriction Categories

Some important definitions on a restriction category $\mathbb{X}$:

- To each object $X \in \mathbb{X}$ is associated the set of **restriction idempotents**,

$$\mathcal{O}(X) := \{e : X \longrightarrow X : \overline{e} = e\}$$

think of these as the "open subsets" of X: though each restriction idempotent is not neccesarily associated to an object of $\mathbb{X}$ (eg., **fdCts** or **fdSmooth**).

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- Given two maps $f, g : X \longrightarrow Y$, write $f \leq g$ if $\overline{f}\, g = f$, and write $f \smile g$ if $\overline{f}\, g = \overline{g}\, f$.

Partial Map Categories

Some restriction categories can be defined as a “category of partial maps”:

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If $\mathbf{C}$ is a category, a **stable system of monics** M is a class of monics which is closed under composition, pullbacks, and contains all isomorphisms.

If we have such a pair $(\mathbf{C}, M)$, there is a natural category of partial maps $\mathrm{Par}(\mathbf{C}, M)$ with the same objects, where a morphism is an equivalence class of spans, with one leg a monic from M.

Examples of Partial map categories

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- $\mathbf{cRing}^{\mathrm{op}}$ has an important class of monics given by localizations of rings.
- Any restriction category $\mathbb{X}$ has a canonical partial map category into which it fully and faithfully embeds: given a restriction category $\mathbb{X}$, define $K(\mathbb{X})$ to have
  - objects (X, e) for $e \in \mathcal{O}(X)$;
  - a map $f : (X, e) \longrightarrow (Y, d)$ is a map $f : X \longrightarrow Y$ such that $\overline{f} = e$ and $ef = d$.

  These have a natural class of monics (the restriction monics).

This category $K(\mathbb{X})$ will be very important!

Join Restriction Categories

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Definition

Let $\mathbb{X}$ be a restriction category. We say that $\mathbb{X}$ is a **join restriction category** if for any family of pairwise compatible maps $(f_i : X \longrightarrow Y)_{i \in I}$, there is a map $\bigvee_{i \in I} f_i : X \longrightarrow Y$ such that

- $\bigvee f_i$ is the join of the f_i's under the partial ordering of maps in a restriction category;
- these joins are compatible with composition: that is, for any $h : Z \longrightarrow X$,

$$h\left(\bigvee_{i \in I} f_i\right) = \bigvee_{i \in I} hf_i.$$

Atlases

Now we can give Grandis's definition (slightly modified by Cockett):

Definition

If $\mathbb{X}$ is a join restriction category, an **atlas of objects from** $\mathbb{X}$ consists of a set of objects $X_i \in \mathbb{X}$, together with a series of maps $X_i \xrightarrow{\phi_{ij}} X_j$ such that:

1. $\phi_{ii}\phi_{ij} = \phi_{ij}$;
2. $\phi_{ij}\phi_{jk} \leq \phi_{ik}$;
3. ϕ_{ij} has partial inverse ϕ_{ji}.

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Each map $\phi_{ii} : X_i \longrightarrow X_i$ is a restriction idempotent, and represents the "open subset" of X_i that the chart is using. The maps ϕ_{ij} define how these charts overlap.

Morphisms of Atlases

There is a natural notion of morphism of these:

Definition

If (U_i, ϕ_{ij}) and (V_k, ψ_{kh}) are atlases of $\mathbf{X}$, then an atlas morphism A consists of a family of maps $U_i \xrightarrow{A_{ik}} V_k$ such that:

1. $\phi_{ii} A_{ik} = A_{ik}$;
2. $\phi_{ij} A_{jk} \leq A_{ik}$;
3. $A_{ik} \psi_{kh} = \overline{A_{ik}}\, A_{ih}$

The last condition ensures the maps glue together correctly on the overlap of charts.

Restriction Category of Atlases

Composition of atlas morphisms is given by using joins:

$$(AB)_{im} := \bigvee_k A_{ik} B_{km}.$$

For any join restriction category $\mathbb{X}$, we then have a join restriction category $\mathsf{Atl}(X)$, with objects atlases, and morphisms atlas morphisms. For example:

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- $\mathsf{Atl}(\mathbf{fdCts})$ "is" real topological manifolds;
- $\mathsf{Atl}(\mathbf{fdSmooth})$ "is" smooth real manifolds;
- $\mathsf{Atl}(\mathsf{Join}(\mathbf{cRing}^{op}, \mathsf{loc}))$ "is" schemes.

Advantages and Disadvantages of Grandis' Manifolds

Advantages:

- works in virtually any context with a reasonable notion of partial map;
- focuses attention on the modelling spaces themselves;
- doesn't live anywhere: no particular preference for a manifold to be a top. space, or a sheaf, or a diffeological space, etc.

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Can we connect atlases up with manifolds which do "live somewhere"?

Gluings of Atlases

We will want settings in which an atlas can be represented by an object.

Definition

Let (X_i, ϕ_{ij}) be an atlas in $\mathbb{X}$. An object $G \in \mathbb{X}$ is said to be the **gluing** of the atlas if there is an atlas morphism $g : (X_i, \phi_{ij}) \longrightarrow (G, 1_G)$ such that:

- each g_i is a restricted isomorphism;
- each $\phi_{i,j} = g_i g_j^{-1}$;
- $1_G = \bigvee_i \overline{g_i^{-1}}$.

Say that a join restriction category **has gluings** in case there is a gluing for every atlas.

For example, **top** and **loc** have all gluings of atlases. Clearly, **fdCts** and **fdSmooth** do not.

The question

Given a join restriction category $\mathbb{X}$, is there a category of "spaces" $\mathbb{S}$ such that:

- **there is a a restriction and join preserving full and faithfully embedding of $\mathbb{X}$ into $\mathbb{S}$;**
- **$\mathbb{S}$ has all gluings?**

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We could then describe $\mathbb{X}$-atlases as those objects of $\mathbb{S}$ which are the gluing of an $\mathbb{X}$-atlas, as is done say for schemes. This gives an alternative view of these objects which doesn't involve atlases.

Locales?

One thing which will not work in general is to use topological spaces (more specifically, locales). It is true that any join restriction category $\mathbb{X}$ has a restriction join-preserving functor

$$\mathcal{O} : \mathbb{X} \longrightarrow \mathbf{loc}$$

$$X \mapsto \mathcal{O}(X)$$

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$$\mathcal{O}(f)(d) = \overline{fd}$$

But this functor is in general neither faithful nor full.
Think of schemes: the Zariski topology (which is what the above is for affine schemes) is not enough information!

C-sheaves

Thinking of schemes, in addition to assigning a locale to each $X \in \mathbb{X}$, we need additional data: we need to assign a sheaf to each locale. But with values in which category? For now, we work with an arbitrary category with coproducts.

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Definition

Let **C** be a category with all small coproducts. Define a restriction category **C**-sheaves where:

- an object is a locale L, together with a (covariant) functor $F : L \longrightarrow \mathbf{C}$ which satisfies the (covariant) sheaf condition;
- a map from (L, F) to (M, G) partial locale map $f : M \longrightarrow L$, as well as a natural transformation $\alpha : G \longrightarrow fF$;
- there is a natural restriction structure on the arrows.

C-sheaves have gluings

We then have the following results:

Proposition

- *If* **C** *has all small coproducts,* **C***-sheaves is a join restriction category.*
- *If* **C** *has all small colimits,* **C***-sheaves has all gluings.*

The Generalized Spec Embedding

But which category **C** should we use? Given an object $X \in \mathbb{X}$, and a restriction idempotent $e : X \longrightarrow X$, we need to assign e to some object. Since this is completely arbitrary, there is really only one choice: $e \mapsto (X, e) \in K(\mathbb{X})$.

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Theorem

For any join restriction category $\mathbb{X}$, there is a faithful join-preserving restriction functor

$$\mathbb{X} \longrightarrow K(\mathbb{X})\text{-sheaves}$$

which sends X to $\mathcal{O}(X)$, with the sheaf that assigns $e \in O(X)$ to $(X, e) \in K(X)$.

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For affine schemes, this is exactly the usual spec embedding of affine schemes into ringed spaces! (In this case, $K(X) = \mathrm{cRing}^{\mathrm{op}}$, so covariant sheaves become usual (contravariant) sheaves on cRing).

Is the spec embedding full?

However:

- In general this embedding is not full.
- For example, for affine schemes, one specializes to *locally* ringed spaces and *locally* ringed morphisms.
- So the general theory of join restriction categories gives you a candidate category: to make the embedding full, you have to work a bit with that particular category.
- But it appears that this specialization is possible in many settings of interest.

Smooth Manifolds again

By identifying smooth manifolds with their algebras of smooth maps to $\mathcal{R}$, a similar construction sees smooth manifolds as sheaves on $\mathcal{R}$-algebras which locally look like $\mathcal{R}^n$.
But there is another concept of space which smooth manifolds live inside: diffeological spaces.

Diffeological Spaces

Invented by Souriau in the late 1970's, greatly expanded by Patrick Iglesias-Zemmour in an (ongoing) book "Diffeology" (available on his website).

Definition

A **diffeological space** is a set X, together with, for any n and U an open subset of $\mathcal{R}^n$, a set of functions $U \longrightarrow X$ called "plots", such that:

- constant functions from the one point set are plots;
- if $f : U \longrightarrow V$ is a smooth function, where U is an open subset of $\mathcal{R}^n$ and V an open subset of $\mathcal{R}^m$, and $P : V \longrightarrow X$ is a plot, then fP is also a plot;
- if $P : U \longrightarrow X$ is a function such that $U = \bigcup_i U_I$ and each P_{U_i} is a plot, then P is a plot.

A map between diffeological spaces is one that sends plots to plots.

Uses of Diffeological Spaces

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The second point is why they are gaining some attention (see a recent paper by Alan Weinstein and others, "Groupoid Symmetry And Constraints In General Relativity. 1: Kinematics"). However, the first point is the one which is important to us: one can see smooth manifolds as diffeological spaces which locally look like $\mathcal{R}^n$. Does the diffeological construction generalize?

Diffeological Spaces as Concrete Sheaves

Recently (2010) Baez and Hoffnung demonstrated the following result.

- Let **C** be the category where an object is an open subset of some $\mathcal{R}^n$, and a map is a smooth function.
- This category has a natural site associated to it: the covering families of $U \subseteq \mathcal{R}^n$ are those $U_i \subseteq \mathcal{R}^n$ such that $\bigcup_i U_i = U$.

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- This category has a natural site associated to it: the covering families of $U \subseteq \mathcal{R}^n$ are those $U_i \subseteq \mathcal{R}^n$ such that $\bigcup_i U_i = U$.
- Then diffeological spaces are the "concrete sheaves" on this "concrete site".
- This site is "subcanonical", i.e., every representable is a sheaf.

The diffeological construction for a join restriction category

Notice that $\mathbf{C} = K(\mathbf{fdSmooth})$! In fact, the above idea works generally for any join restriction category:

Proposition

If $\mathbb{X}$ is any join restriction category, then there is a natural site associated to $K(\mathbb{X})$, with the covering families of (X, e) being those (X, e_i) such that $\bigvee e_i = e$. Moreover, this site is subcanonical.

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Thus, another natural choice for a "large category of spaces" associated to a join restriction category $\mathbb{X}$ is $\mathrm{Sh}(K(\mathbb{X}))$ (our sites will not in general be concrete, so we cannot ask for concrete sheaves).

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- if $\mathbb{X}$ has joins, this embedding is actually into sheaves;
- for any Grothendieck topos, its restriction category of all monics has all gluings.

So again we have a faithful embedding of $\mathbb{X}$ into a "category of spaces" (a sheaf category) which has all gluings.

A Faithful and Full Embedding?

But I believe we can do better!

- Cockett and Lack showed that the faithful embedding of $\mathsf{Par}(K(\mathbb{X}), \text{res. monics})$ into $\mathsf{Par}(\mathsf{PSh}(K(\mathbb{X}), \text{all monics})$ can always be made full by restricting the monics appropriately.

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- I believe this partial map category also has all gluings.

Which would give the desired result: a full and faithful embedding of $\mathbb{X}$ into a category of spaces (again, a sheaf category).

Conclusion

Given a join restriction category $\mathbb{X}$, there are two possibilities for "large categories of spaces" which atlases of $\mathbb{X}$ live inside:

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Given a join restriction category $\mathbb{X}$, there are two possibilities for "large categories of spaces" which atlases of $\mathbb{X}$ live inside:

- the scheme approach: covariant sheaves from a locale to $K(X)$;
- the diffeology approach: contravariant sheaves from the canonical site on $K(X)$ to **set**.

Applications

Some applications:

- What do shemes look like with the diffeology (Yoneda) embedding? This should give (another) definition of schemes.

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- Does either version of space make smooth manifolds modelled on convenient vector spaces (currently using atlases) easier to work with?

Without the formalism of join restriction categories, these connections would be hard to see.