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Towards cotangent categories

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Based on joint work with J.S. Lemay

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Introduction	Tangent categories	Linear dualization and the cotangent bundle	Conclusion
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Overview			

Tangent categories are a *minimal* categorical setting for differential geometry.

- Tangent categories span a wide variety of examples from differential geometry and algebraic geometry to abstract homotopy theory (functor calculus).
- Many structures can be defined in a tangent category, including vector bundles, differential forms, and connections.
- In this talk the focus is on how to define and work with the *cotangent* bundle in tangent categories.

Plan:

- Review of tangent categories and differential bundles.
- Oefine *linear dualization* in a tangent category and how this gives a cotangent bundle.
- Son we axiomatically define a *cotangent category*?

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langent	category definition	on	

Tangent categories abstract the structure of the tangent bundle functor on the category of smooth manifolds.

Definition (Rosický 1984, modified Cockett/Cruttwell 2014)

A tangent category consists of a category X with:

- an endofunctor $T : \mathbb{X} \to \mathbb{X}$;
- a natural transformation $p: T \rightarrow 1_{\mathbb{X}}$;
- for each M, the pullback of n copies of $p_M : TM \to M$ along itself exists (and is preserved by each T^m), call this pullback T_nM ;
- for each M ∈ X, p_M : TM → M has the structure of a commutative monoid in the slice category X/M, in particular there are natural transformations + : T₂ → T, 0 : 1_X → T;

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Tangent of	category definition	(continued)	

Definition

- (canonical flip) there is a natural transformation $c : T^2 \rightarrow T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \rightarrow T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for ℓ and c;
- (tangent spaces have trivial tangent bundle) the following is a pullback:

$$\begin{array}{cccc}
T_2 M & \xrightarrow{\nu} & T^2 M \\
 \pi_0 p_M & & & \downarrow \\
 m_{-0_M} & & & \downarrow \\
 M & \xrightarrow{0_M} & TM \\
\end{array}$$

where $\nu = \langle \pi_0 0_M, \pi_1 \ell \rangle T(+)$.

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Examples			

- Smooth manifolds with their tangent bundle.
- Convenient manifolds (a certain type of infinite-dimensional manifold) with their *kinematic* tangent bundle.
- The infinitesimally linear objects in a model of synthetic differential geometry (SDG)
- The category of C^{∞} -rings.
- Commutative ri(n)gs and its opposite, as well as various other categories in algebraic geometry.
- (MacAdam) The category of all small categories with finite limits is a tangent category, where

T(X) = Beck modules in X (Abelian group objects in X)

- Abelian functor calculus gives a tangent category, and Goodwillie functor calculus gives an (infinity) tangent category.
- The vector fields in any tangent category form a new tangent category (as do many other constructions).

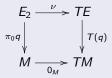
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Differential bundles

The analog of vector bundles in a tangent category are:

Definition (Cockett/Cruttwell 2015)

A differential bundle in a tangent category consists of an additive bundle $(q: E \rightarrow M, \sigma, \zeta)$ and a map $\lambda: E \rightarrow TE$ such that the following is a pullback:



where E_2 is the pullback of q along itself, and $\nu = \langle \pi_0 0_E, \pi_1 \lambda \rangle T(\sigma)$.

A differential bundle E over 1 has

$$TE \cong E \times E.$$

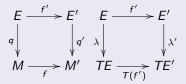
(MacAdam) In the tangent category of smooth manifolds, differential bundles = vector bundles (with their local triviality condition!), (=, =)

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Differential bundles continued

Definition

A linear morphism of differential bundles from $(q : E \to M, \sigma, \zeta, \lambda)$ to $(q' : E' \to M', \sigma', \zeta', \lambda')$ consists of a morphism in the arrow category (f, f') which preserve the λ 's:



call the associated category Lin(X).

- In the tangent category of smooth manifolds, morphisms of differential bundles = linear bundle morphisms.
- With some additional pullback assumptions, Lin(X) is a fibration over the base category X.

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• T can be seen as a functor $\mathbb{X} \to \text{Lin}(\mathbb{X})$.

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Cotanger	nt bundle?		

Recall that the cotangent bundle T^*M is defined by taking the *dual* of each of the tangent spaces of M.

- The cotangent bundle *is not* an endofunctor on the category of smooth manifolds.
- It does give an endofunctor when restricted to etale maps...but not general smooth maps.
- It does give a functor to a different category, though: the *dual* of the fibration of differential/vector bundles.

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The dual	fibration		

Any fibration $F : \mathbb{A} \to \mathbb{B}$ has an associated **dual** fibration given by taking the opposite category in each fibre. For example:

• Consider the simple fibration over a Cartesian category which has objects pairs (A, A') with maps $(f, f') : (A, A') \rightarrow (B, B')$ such that

 $f: A \rightarrow B$ and $f': A \times A' \rightarrow B'$

• Its dual fibration has objects pairs (A, A') with maps $(f, f^*) : (A, A') \rightarrow (B, B')$ such that

$$f: A \rightarrow B$$
 and $f^*: A \times B' \rightarrow A'$.

• This is also known as the category of *lenses*, and has appeared in many places (database theory, functional programming, dialectica categories, machine learning): its morphisms have a very useful *bidirectional* nature.

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More dual fibrations

• Recall that the arrow category/codomain fibration of a category X with pullbacks has objects maps $q: E \to M$ and morphisms pairs (f, f') which give commuting squares



 Its dual fibration again has objects maps q : E → M but now a morphism is a pair (f, f*) with

$$f: M \to M'$$
 and $f^*: E' \times_{M'} M \to E$

(This is sometimes called the category of **dependent** lenses: notice it again has a bidirectional nature!)

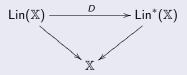
 The dual of the fibration Lin(X), Lin*(X) is the same as above, except now the objects are differential bundles, and f* must be linear.

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Linear dualization

Definition

If $\mathbb X$ is a tangent category, a **linear dualization on** $\mathbb X$ consists of a fibration functor (ie., it preserves Cartesian arrows)



which is compatible with the tangent structure, that is

$$\begin{array}{c|c} \mathsf{Lin}(\mathbb{X}) & \stackrel{D}{\longrightarrow} \mathsf{Lin}^{*}(\mathbb{X}) \\ \bar{\tau} & & & \\ \bar{\tau} & & & \\ \mathsf{Lin}(\mathbb{X}) & \stackrel{D}{\longrightarrow} \mathsf{Lin}^{*}(\mathbb{X}) \end{array}$$

commutes (where \overline{T} and \widetilde{T} are functors induced by T).

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Cotangent bundle functor

In any tangent category X with a linear dualization D, we get an associated functor T^* by composing T with D:

$$\mathbb{X} \xrightarrow{T} \mathsf{Lin}(\mathbb{X}) \xrightarrow{D} \mathsf{Lin}^*(\mathbb{X}).$$

This assigns to each M its "cotangent bundle" $q: T^*M \to M$, and assigns to a map $f: M \to N$ its "pullback" (as it is called in differential geometry)

$$T^*(f): T^*N \times_N M \to T^*M$$

(in other terminology, it assigns to each f a dependent lens).

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Pullback of covector fields

For the next few slides, we'll assume $\mathbb X$ is a tangent category with a linear dualization.

Definition

Define a **covector field on** M in \mathbb{X} to be a map $\omega : M \to T^*(M)$ which is a section of $q : T^*(M) \to M$.

It is a standard result that covector fields can be "pulled back", and this also holds in our setting:

Lemma

If $f: M \to N$ is a map and $\omega: N \to T^*(N)$ is a covector field on N, then

$$M \xrightarrow{\langle f\omega, 1 \rangle} T^*(N) \times_N M \xrightarrow{T^*(f)} T^*(M)$$

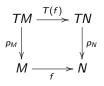
is a covector field on M.

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Cotangent bundle functor on etale maps

We also get an induced endofunctor on the base category when restricted to etale maps maps:

- In any tangent category, define a **etale** map to be a map f: M
 - \rightarrow N such that



is a pullback (this agrees with the usual notion in smooth manifolds).

- That is, (f, T(f)) is a Cartesian arrow in the fibration Lin(X), so D of it is a Cartesian arrow in Lin*(X)
- But Cartesian arrows in a dual fibration correspond to Cartesian arrows in the original fibration, so applying the above to an etale map gives a morphism of Lin(X), and taking the top component of this map gives an endofunctor

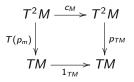
$$T^*: etale(\mathbb{X}) \to etale(\mathbb{X}).$$

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Some other structure on the cotangent bundle

There are some other results one gets in this abstract setup:

• The canonical flip $c: T^2
ightarrow T^2$ gives linear isomorphisms



Applying D to this gives isomorphisms

$$T(T^*M)\cong T^*(TM)$$

• Can build the Liouville-Hamilton vector field on the cotangent bundle: a vector field on T^*M , ie., a map

$$T^*(M) \to T(T^*(M))$$

• Can build the canonical covector field on the cotangent bundle, ie., a map

$$T^*(M) \to T^*(T^*(M))$$

Cotangent categories?

Can we directly axiomatize a "category equipped with a cotangent bundle"?

- Why? In many places the cotangent bundle is seen as a more natural structure than the tangent bundle, so it might be nice to axiomatize it directly.
- $\bullet\,$ We could define a cotangent category as a category $\mathbb X$ equipped with a functor

$$T^*: \mathbb{X} \to \mathsf{AdBun}^*(\mathbb{X})$$

where AdBun(X) is the category of additive bundles in X.

• Some of the other structure is less clear, though: for example, as we saw on the previous slide, the canonical flip gives an isomorphism

$$T(T^*M) \cong T^*(TM)$$

not an isomorphism $T^*(T^*(M)) \cong T^*(T^*(M))$.

• Also not clear how to define the analog of differential bundles directly in the case cotangent structure.

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Conclusions					

In conclusion:

- A tangent category equipped with a *linear dualization* functor has an associated cotangent bundle functor, and the technology of the dual fibration is very helpful in defining this abstractly.
- Some standard results about the cotangent bundle can be recovered from this abstract setup (and some non-standard results immediately follow as well).
- It is not yet clear (to us) how to define a cotangent category, but we're still thinking about it.

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