# Connections in tangent categories

Geoff Cruttwell
Mount Allison University
(joint work with Robin Cockett)

Union College Mathematics Conference
Union College, October 20th, 2013

# Tangent category definition

**Definition (Rosicky 1984, modified Cockett/Cruttwell 2013)**

A **tangent category** consists of a category $\mathbb{X}$ with:

- an endofunctor $\mathbb{X} \xrightarrow{T} \mathbb{X}$;
- a natural transformation $T \xrightarrow{p} I$;
- for each $M$, the pullback of $n$ copies of $TM \xrightarrow{p_M} M$ along itself exists (and is preserved by $T$), call this pullback $T_n M$;
- such that for each $M \in \mathbb{X}$, $TM \xrightarrow{p_M} M$ has the structure of a commutative monoid in the slice category $\mathbb{X}/M$, in particular there are natural transformation $T_2 \xrightarrow{+} T$, $I \xrightarrow{0} T$;

# Tangent category definition continued...

## Definition

- (canonical flip) there is a natural transformation $c : T^2 \longrightarrow T^2$ which preserves additive bundle structure and satisfies $c^2 = 1$;
- (vertical lift) there is a natural transformation $\ell : T \longrightarrow T^2$ which preserves additive bundle structure and satisfies $\ell c = \ell$;
- various other coherence equations for $\ell$ and $c$;
- (universality of vertical lift) the following is a pullback diagram:

$$\begin{array}{ccc} T_2(M) & \xrightarrow{v:=\langle \pi_0 \ell, \pi_1 0_T \rangle T(+)} & T^2(M) \\ {\scriptstyle \pi_0 p = \pi_1 p} \downarrow & & \downarrow {\scriptstyle T(p)} \\ M & \xrightarrow[0]{} & T(M) \end{array}$$

# Examples

*(i)* Finite dimensional smooth manifolds with the usual tangent bundle structure.

*(ii)* Convenient manifolds with the kinematic tangent bundle.

*(iii)* Any Cartesian differential category is a tangent category, with $T(A) = A \times A$ and $T(f) = \langle Df, \pi_1 f \rangle$.

*(iv)* The infinitesimally linear objects in any model of synthetic differential geometry.

*(v)* Both commutative ri(n)gs and its opposite category have tangent structure.

*(vi)* The category of C-$\infty$-rings has tangent structure.

# Some theory of tangent categories

*(i)* A **vector field** on $M$ is a map $X : M \longrightarrow TM$ which is a section of $p : TM \longrightarrow M$.

*(ii)* These vector fields have a Lie bracket operation $[X, Y]$ which satisfies the usual properties of a bracketing operation.

*(iii)* The "tangent spaces" of a tangent category form a Cartesian differential category.

*(iv)* $T$ is automatically a monad.

*(v)* A tangent category in which $T$ is representable has a commutative rig $R$ with $R^D \cong R \times R$ (ie., it satisfies the "Kock-Lawvere" axiom).

# Differential bundles

## Definition

A **differential bundle** in a tangent category consists of an additive bundle $q : E \longrightarrow M$ with a map $\lambda : E \longrightarrow TE$ such that

- all pullbacks along $q$ exist and are preserved by $T$;
- $(\lambda, 0)$ and $(\lambda, \zeta)$ are additive bundle morphisms;
- the following is a pullback diagram:

$$\begin{array}{ccc} E_2 & \xrightarrow{\mu := \langle \pi_0 \lambda, \pi_1 0 \rangle T(\sigma)} & T(E) \\ {\scriptstyle \pi_0 q = \pi_1 q}\downarrow & & \downarrow{\scriptstyle T(q)} \\ M & \xrightarrow[0]{} & T(M) \end{array}$$

  where $E_2$ is the pullback of $q$ along itself;
- $\lambda \ell_E = \lambda T(\lambda)$.

# Examples and properties

(i) Any object has an associated "trivial" differential bundle $1_M = (1_M, 1_M, 1_M, 0_M)$.

(ii) The tangent bundle of each object $M$, $p : TM \longrightarrow M$ is a differential bundle.

(iii) The pullback of a differential bundle along any map is a differential bundle.

(iv) If $q : E \longrightarrow M$ is a differential bundle, so is $Tq : TE \longrightarrow TM$.

(v) Each fibre over a point $E_a M$ is a "vector space", ie., $T(E_a M) \cong E_a M \times E_a M$.

# Differential bundle morphisms

- A **morphism of differential bundles** between differential bundles $(q : E \longrightarrow M)$, $(q' : E' \longrightarrow M')$ is simply a pair of maps $f : E \longrightarrow E'$, $g : M \longrightarrow M'$ making the obvious diagram commute.
- A morphism of differential bundles $(f, g)$ is **linear** if it also preserves the lift, that is,

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda'} \\ T(E) & \xrightarrow[\quad T(f) \quad]{} & T(E') \end{array}$$

  commutes.

(This corresponds to the ordinary definition of linear morphisms between vector bundles in the canonical example).

# What are connections?

Intuitive idea: can "move tangent vectors between different tangent spaces". Moving a tangent vector around a closed curve measures the "curvature" of the space. But how to precisely express what a connection is? Some answers:

- as a "horizontal subspace";
- as a "connection map";
- as a notion of "parallel tranport";
- as a "covariant derivative".

# What are connections?

Intuitive idea: can "move tangent vectors between different tangent spaces". Moving a tangent vector around a closed curve measures the "curvature" of the space. But how to precisely express what a connection is? Some answers:

- as a "horizontal subspace";
- as a "connection map";
- as a notion of "parallel tranport";
- as a "covariant derivative".

Quoting Spivak:

> *"I personally feel that the next person to propose a new definition of a connection should be summarily executed."*

# Claim

I claim that:

- Connections have a very natural expression in terms of the lift map for differential bundles.
- The canonical flip map $c$ gives a natural and easy way to express the properties of being “flat” or “torsion-free”.

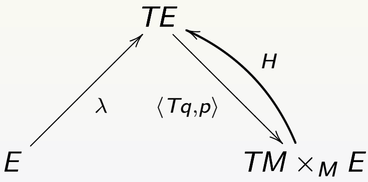
# Two fundamental maps

A differential bundle has two key maps involving $TE$ whose composite is the zero map:

$$E \xrightarrow{\lambda} TE \xrightarrow{\langle Tq, p \rangle} TM \times_M E$$

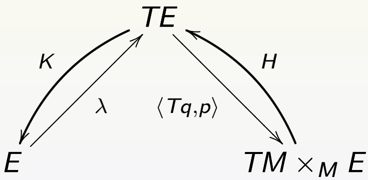
# Horizontal lift

A **connection** consists of a linear section of $H$ of $\langle Tq, p\rangle$ called the **horizontal lift**...



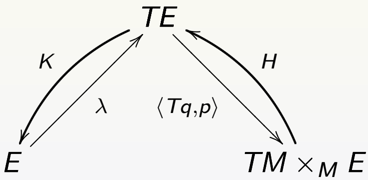
# Connector

which in addition has a linear retraction $K$ of $\lambda$ called the **connector**:



# Connection definition

that satisfies the equations $HK = 0$ and $(\lambda K \oplus p0) + \langle T(q), p\rangle H = 1$.



# Connections in a tangent category

Complete definition:

**Definition**

A **connection** on a differential bundle $q : E \longrightarrow M$ consists of:

- a linear section $K$ of $\lambda$;
- a linear retraction $H$ of $\langle T(q), p \rangle$;
- such that $HK = 0$ and $(\lambda K \oplus p0) + \langle T(q), p \rangle H = 1$.

A connection on the tangent bundle $p : TM \longrightarrow M$ is called an **affine connection**.

**Proposition**

*If a differential bundle $q$ has a connection $(K, H)$ then $TE$ is the pullback (over $M$) of $TM$ and two copies of $E$.*

# Canonical examples

Any differential object $A$ (Cartesian spaces in the standard example) is a differential bundle over 1 and for these one can define:

- $K : TA \longrightarrow A$ by $K(v, a) := v$ and
- $H : A \longrightarrow TA$ by $H(a) := (0, a)$.

# Canonical examples

Any differential object $A$ (Cartesian spaces in the standard example) is a differential bundle over 1 and for these one can define:

- $K : TA \longrightarrow A$ by $K(v, a) := v$ and
- $H : A \longrightarrow TA$ by $H(a) := (0, a)$.

The tangent bundle of any differential object $A$ is also a differential bundle $p : TA \longrightarrow A$ with a canonical (affine) connection:

- $K' : T^2A \longrightarrow TA$ by $K(d, v, w, a) := (d, a)$ and
- $H' : A \times A \times A \longrightarrow T^2A$ by $H(v, w, a) := (0, v, w, a)$.

# $K$ from $H$

## Proposition

*Suppose* $(\mathbb{X}, \mathbb{T})$ *is a tangent category with negatives, and* $H$ *is a section of* $\langle T(q), p\rangle$ *on a differential bundle* q. *Then the pair* $(\{1 - \langle Tq, p\rangle H\}, H)$ *is a connection on* q.

Note that this requires negatives! It also uses the universal property of $\lambda$.

# $H$ from $K$

**Proposition**

*Let $(\mathbb{X}, \mathbb{T})$ be a tangent category,* q *a differential bundle, and $K$ a connector on* q*. If* q *has a section $J$ of $\langle T(q), p\rangle$, then the pair $(K, J(1-(\lambda K \oplus p0))$ is a connection on* q*.*

This also requires negatives, but also needs $\langle T(q), p\rangle$ to have at least one section $J$ (the resulting connection is independent of the choice of such $J$).

# Covariant derivative

For a differential bundle q, let $\chi(\mathsf{q})$ denote the set of sections of $q$.

# Covariant derivative

For a differential bundle q, let $\chi$(q) denote the set of sections of $q$.

## Definition

Let $(K, H)$ be a connection on q. Its **covariant derivative** is an operation

$$\nabla K : \chi(\mathrm{p}) \times \chi(\mathrm{q}) \longrightarrow \chi(\mathrm{q})$$

given by mapping $(w : M \longrightarrow TM, s : M \longrightarrow E)$ to

$$\nabla K(w, s) := M \xrightarrow{w} TM \xrightarrow{T(s)} TE \xrightarrow{K} E$$

(This corresponds to one of the definitions of connection in the literature).

# Flat connections

The definition of a connection being flat in the literature is quite complicated, but by using the map $c$ we can make a very simple definition:

# Flat connections

The definition of a connection being flat in the literature is quite complicated, but by using the map $c$ we can make a very simple definition:

**Definition**

Say that a connection is **flat** if $cT(K)K = T(K)K$.

This does correspond to the usual definition:

# Curvature

## Definition

For a tangent category with negatives, the **curvature** of a connector $K$ on q is the function

$$F : \chi(M) \times \chi(M) \times \chi(E) \longrightarrow \chi(E)$$

given by mapping $(w_1 : M \longrightarrow TM, w_2 : M \longrightarrow TM, s : M \longrightarrow E)$ to

$$FK(w_1, w_2, s) := \nabla(w_1, \nabla(w_2, s)) - \nabla(w_2, \nabla(w_1, s)) - \nabla([w_1, w_2], s).$$

(Where the bracketing operation above is the abstract Lie bracket in tangent categories).

## Theorem

*If $(K, H)$ is a flat connection then its curvature is identically 0.*

# Torsion-free connections

Torsion-free connections are connections on the tangent bundle for which the movement of tangent vectors does not "twist". Again there is a simple definition of this in our setting:

**Definition**

Say that a connection on a tangent bundle $p : TM \longrightarrow M$ is **torsion-free** if $cK = K$.

# Torsion-free connections

Torsion-free connections are connections on the tangent bundle for which the movement of tangent vectors does not “twist”. Again there is a simple definition of this in our setting:

**Definition**

Say that a connection on a tangent bundle $p : TM \longrightarrow M$ is **torsion-free** if $cK = K$.

This does correspond to the usual definition:

**Theorem**

*If $(K, H)$ is a torsion-free connection with associated covariant derivative $\nabla$ then*

$$[w_1, w_2] - \nabla(w_1, w_2) - \nabla(w_2, w_1)$$

*is identically zero.*

# Conclusions

To sum up:

- Connections can be defined in tangent categories in a way that makes natural use of the lifting map $\lambda$.
- Flat and torsion-free connections can be defined in tangent categories in a way that makes natural use of the map $c$.
- In special cases, our definition of connection is equivalent to the usual one(s).
- The way presented here is perhaps the most natural, categorically.

# Future work

- What do connections look like in the different tangent categories? In particular, does it help with understanding connections in situations without negatives (eg., tropical geometry)?
- Can we define de Rham cohomology of vector bundles with a connection?
- How does this fit with Rory Lucyshyn-Wright's theory of integration?

# Future work

- What do connections look like in the different tangent categories? In particular, does it help with understanding connections in situations without negatives (eg., tropical geometry)?
- Can we define de Rham cohomology of vector bundles with a connection?
- How does this fit with Rory Lucyshyn-Wright's theory of integration?

References:


- Cockett, R. and Cruttwell, G. Differential structure, tangent structure, and SDG. To appear in *Applied Categorical Structures*, preprint available at

  `http://www.mta.ca/~gcruttwell/publications/sman3.pdf`
- Rosický, J. Abstract tangent functors. *Diagrammes*, 12, Exp. No. 3, 1984.