3) In the presentation of M, we have

$$Tor(M) \cong \mathcal{R}/\langle a_1\rangle \oplus \cdots \oplus \mathcal{R}/\langle a_m\rangle$$

In particular M is a torsion module $\iff$ $r = 0$ and in this case $Ann(M) = \langle a_m\rangle$.

Proof.

1) Since M is finitely generated, pick a generating set of M with the minimal cardinality, say $\{x_1, x_2, \cdots, x_n\}$. Let $\mathcal{R}^n$ be a free module of rank n with basis $\{b_1, b_2, \cdots, b_n\}$. Define a map

$$\begin{array}{rccc} \varphi: & \mathcal{R}^n & \to & M \\ & b_i & \mapsto & x_i \end{array}$$

and extend it homomorphically over all of $\mathcal{R}^n$ considered as a $\mathcal{R}$ module. Then φ is a $\mathcal{R}$ module homomorphism. Moreover, given an arbitrary element of M, $\sum_{i=1}^n a_i x_i$, we have

$$\varphi\left(\sum_{i=1}^n a_i b_i\right) = \sum_{i=1}^n a_i x_i$$

so φ is surjective. By Theorem 11 (1st Isomorphism Theorem),

$$\mathcal{R}^n/\ker(\varphi) \cong M$$

Since $\ker(\varphi)$ a submodule of $\mathcal{R}^n$, there exists a basis $y_1, y_2, \cdots, y_m$ of $\mathcal{R}^n$ and elements $a_1, a_2, \cdots, a_m \in \mathcal{R}$ such that $a_1y_1, a_2y_2, \cdots, a_my_m$ is a basis of $\ker(\varphi)$ and $a_i|a_{i+1}$ for $i \in \{1, 2, \cdots, m-1\}$. Therefore

$$M \cong (\mathcal{R}y_1 \oplus \mathcal{R}y_2 \oplus \cdots \oplus \mathcal{R}y_n)/(\mathcal{R}a_1y_1 \oplus \mathcal{R}a_2y_2 \cdots \oplus \mathcal{R}a_my_m)$$

Consider the natural sujrective $\mathcal{R}$ module projection:

$$\begin{array}{rccl} \pi: & \mathcal{R}y_1 \oplus \mathcal{R}y_2 \oplus \cdots \oplus \mathcal{R}y_n & \to & \mathcal{R}/\langle a_1\rangle \oplus \mathcal{R}/\langle a_2\rangle \cdots \oplus \mathcal{R}/\langle a_m\rangle \oplus \mathcal{R}^{n-m} \\ & (u_1y_1, u_2y_2, \cdots, u_ny_n) & \mapsto & (u_1 + \langle a_1\rangle, u_2 + \langle a_2\rangle, \cdots, \\ & & & u_m + \langle a_m\rangle, (u_{m+1}, \cdots, u_n)) \end{array}$$