Proposition 12.7. *Any conformal map of $D(0,1)$ onto itself is a fractional linear transformation of the form $T(z) = e^{i\theta}\dfrac{z - z_0}{1 - \overline{z_0}z}$ mapping z_0 to 0.*

13 Rouche's Theorem

Definitions

Δ_γ **arg** f Let f be analytic on A and let $\gamma \in A$ be a closed curve homotopic to a point and passing through no zero of f. Then

$$\Delta_\gamma \mathrm{arg} f = 2\pi I(f \circ \gamma, 0)$$

Facts

Proposition 13.1 (Root-Pole Counting Theorem)**.** *Let f be analytic on a region A except for poles $b_1, \cdots, b_m$ and zeroes $a_1, \cdots, a_n$ counted with multiplicity (i.e. u pole of order k, then u repeated in the list k times). Let γ be a closed curve homotopic to a point in A passing through none of a_j or b_l. Then*

$$\int_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i\left[\sum_{j=1}^{n} I(\gamma, a_j) - \sum_{i=1}^{m} I(\gamma, b_i)\right]$$

Proposition 13.2 (Argument Principle)**.** *Let f be analytic on a region A except for poles $b_1, \cdots, b_m$ and zeroes $a_1, \cdots, a_n$ counted with multiplicity. Let γ be a closed curve homotopic to a point and passing through none of a_j or b_ℓ. Then*

$$\Delta_\gamma arg f = 2\pi\left[\sum_{j=1}^{n} I(\gamma, a_j) - \sum_{i=1}^{m} I(\gamma, b_i)\right]$$

Proposition 13.3 (Rouche's Theorem)**.** *(*) Let f and g be analytic on a region A except for a finite number of zeroes and poles in A. Let γ be a closed curve in A homotopic to a point and passing through no zero or pole of f or g. Suppose that on γ, $|f(z) - g(z)| < |f(z)|$. Then*

(i) Δ_γ *arg* f $= \Delta_\gamma$ *arg* g

(ii) $Z_f - P_f = Z_g - P_g$ *where* Z_f *is given by* $Z_f = \sum_{j=1}^{n} I(\gamma, a_j)$ *for* a_j *zeroes of* f *counted with multiplicity,* P_f, Z_g, P_g *defined similarly.*