Proposition 3.5.5. *Let $\mathfrak{X}$ be a normed space and $\mathfrak{M}$ a hyperplane in $\mathfrak{X}$. Then either $\mathfrak{M}$ is closed or $\mathfrak{M}$ is dense in $\mathfrak{X}$.*

Proof. $\overline{\mathfrak{M}}$ is a subspace of $\mathfrak{X}$ such that $\mathfrak{M} \subseteq \overline{\mathfrak{M}} \subseteq \mathfrak{X}$. Since $\dim\left(\mathfrak{X}/\mathfrak{M}\right) = 1$, we have $Q(\overline{\mathfrak{M}}) = \{0\}$ or $\mathfrak{X}/\mathfrak{M}$. If $Q(\overline{\mathfrak{M}}) = \{0\}$, then $\overline{\mathfrak{M}} = \mathfrak{M}$ and so $\mathfrak{M}$ is closed. If $Q(\overline{\mathfrak{M}}) = \mathfrak{X}/\mathfrak{M}$ then $\overline{\mathfrak{M}} = \mathfrak{X}$, so $\mathfrak{M}$ is dense in $\mathfrak{X}$. ∎

Theorem 3.5.6. *Let $\mathfrak{X}$ be a normed space. Let $f : \mathfrak{X} \to \mathbb{F}$ be a linear functional. Then f is continuous $\iff$ $\ker(f)$ is closed.*

Proof.
Suppose $f = 0$. Then $\ker(f) = \mathfrak{X}$ and we are done. Similarly, if $\ker(f) = \mathfrak{X}$, then $f = 0$, which is continuous and we are done. Thus we can suppose $f \neq 0$.
$\Longrightarrow$ Since f is continuous, for any $U \subseteq \mathbb{F}$ open, we have $f^{-1}(U)$ open. Consider $\{0\} \subseteq \mathbb{F}$. This set is closed, so $\mathbb{F} \setminus \{0\}$ open and $f^{-1}(\mathbb{F} \setminus \{0\})$ open. Then

$$\begin{aligned} f^{-1}(\mathbb{F} \setminus \{0\}) &= \{x \in \mathfrak{X} \mid f(x) \in \mathbb{F} \setminus \{0\}\} \\ &= \{x \in \mathfrak{X} \mid f(x) \neq 0\} \\ &= \{x \in \mathfrak{X} \mid x \notin \ker(f)\} \\ &= \mathfrak{X} \setminus \ker(f) \end{aligned}$$

Since $f^{-1}(\mathbb{F} \setminus \{0\})$ open, we have $\mathfrak{X} \setminus \ker(f)$ open, so $\ker(f)$ closed.

$\Longleftarrow$ Assume $\ker(f)$ closed. Let $Q : \mathfrak{X} \to \mathfrak{X}/\ker(f)$ be the canonical quotient map. By Proposition 3.3.4, $||Qx|| \leq ||x||_{\mathfrak{X}}$, and so by Proposition 3.1.10, Q is continuous.

By Proposition 3.5.2, since f a linear functional, $\ker(f)$ a hyperplane. Thus $\dim\left(\mathfrak{X}/\ker(f)\right) = 1$ and so $\mathfrak{X}/\ker(f) \cong \mathbb{F}$. Let φ denote the isomorphism between these two spaces. Since $\dim\left(\mathfrak{X}/\ker(f)\right) = 1$, this space is finite dimensional, and since φ linear, Proposition 3.2.4 gives that φ is continuous.

Thus $\varphi \circ Q : \mathfrak{X} \to \mathbb{F}$ is a continuous, linear functional, with, similar to the proof of Proposition 3.5.2, $\ker(\varphi \circ Q) = \ker(f)$. Moreover, since $f \neq 0$, we have $\varphi \circ Q \neq 0$. By Proposition 3.5.3, there exists some scalar α such that $f = \alpha(\varphi \circ Q)$. Since the scalar multiple of a continuous function is continuous, we have that f is continuous. ∎