## 5.2 Fatou's Lemma

**Theorem 5.5** (Fatou's Lemma)**.** *Suppose* $(X, \mathcal{C}, \mu)$ *a measure space. If* $\{f_n\}_{n\in\mathbb{N}}$ *is a sequence of non-negative,* $\mu$ *measurable functions, then*

$$\int \liminf_{n\in\mathbb{N}}\{f_n\}\,d\mu \le \liminf_{n\in\mathbb{N}} \int f_n\,d\mu$$

*Proof.* Fix $m \in \mathbb{N}$. Let $g_m(x) = \inf\limits_{n\ge m}\{f_n(x)\}$.

Fix $x$. Then $g_m(x) \le f_n(x)$ for $n \ge m$. Therefore

$$\int g_m\,d\mu \le \int f_n\,d\mu \text{ if } n \ge m$$

Therefore

$$\int g_m\,d\mu \le \inf_{n\ge m}\int f_n\,d\mu \le \sup_{k\ge 1}\left(\inf_{n\ge k}\int f_n\,d\mu\right) = \liminf_{n\in\mathbb{N}}\int f_n\,d\mu$$

Since $m$ was arbitrary, this is true for all $m$. Therefore

$$\lim_{m\to\infty}\int g_m\,d\mu \le \liminf_{n\in\mathbb{N}}\int f_n\,d\mu$$

Again fix $x$. Then $g_m(x) \le g_{m+1}(x)$ by rules of infemum (glb) (since $g_{m+1}$ is taking the infemum over a smaller set). Thus $\{g_m\}_{m\in\mathbb{N}}$ is a monotone increasing sequence. We will now apply Theorem 5.4 (Monotone Convergence Theorem). That is:

$$\lim_{m\to\infty}\int g_m\,d\mu = \int \lim_{m\to\infty} g_m\,d\mu = \int \lim_{m\to\infty}\left(\inf_{n\ge m}\{f_n(x)\}\right)\,d\mu = \int \liminf_{m\in\mathbb{N}}\{f_n(x)\}\,d\mu$$

Therefore

$$\int \liminf_{m\in\mathbb{N}}\{f_n(x)\}\,d\mu \le \liminf_{n\in\mathbb{N}}\int f_n\,d\mu$$

□

**Example 5.6.** For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } n \le x \le n+1 \\ 0 & \text{else.} \end{cases}$$