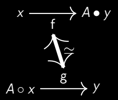


# Quantum Message Passing Logic - Day 2

**FMCS 2023, Sackville**

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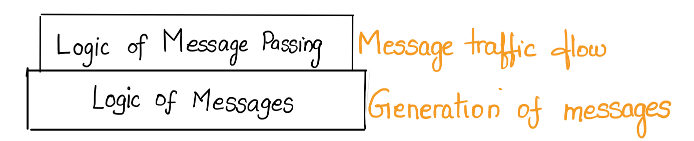
NIST, USA

September 11, 2023

# Recap …

# Message passing logic

Linear actegories are a categorical semantics of the message passing logic.



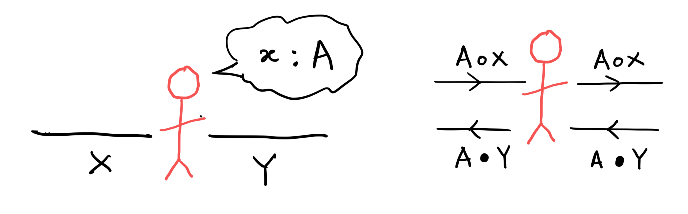
# Linear actegories

Loosely, a linear actegory is a symmetric monoidal category acting on a linearly distributive category on the left and the right.

Let $(\mathbb{A}, *, I)$ be a symmetric monoidal category. A symmetric linear $\mathbb{A}$-actegory consists the following data:

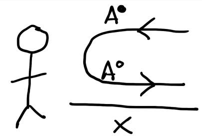
- A symmetric linearly distributive category $(X, \otimes, \top, \oplus, \bot)$
- Functors

$$\circ : \mathbb{A} \times \mathbb{X} \to \mathbb{X} \qquad \bullet : \mathbb{A}^{\mathrm{op}} \times \mathbb{X} \to \mathbb{X}$$

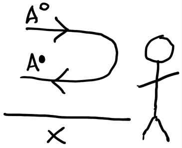


# Linear actegories (cont...)

For all $A \in \mathbb{A}$, $A \circ -$ is left adjoint to $A \bullet -$ :




$\eta_X : X \to A \bullet (A \circ X)$



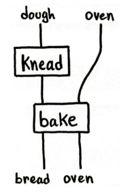

$\epsilon_X : A \circ (A \bullet X) \to X$

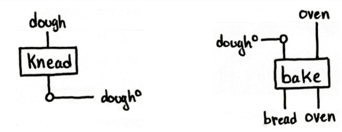
# Linear actegories: an analogy


| | | |
|---:|:---:|:---|
| The network of roads | :: | Linearly distributive cats |
| Market, school, house.. | :: | Monoidal categories |
| Vehicles | :: | Messages |
| Getting vehicles on to the road? | :: | Adjoint functors: $A \circ - \dashv A \bullet -$ |

# Concurrent process histories and resource transducers

Chad Nester (2022)




**Motivation:**

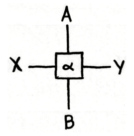
Capture the *movement* of resources or information across different components of a concurrent process

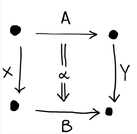
**Methodology:**

- Considers the concurrent process as a resource theory (strict symmetric monoidal category)
- Augments the string diagrams for symmetric monoidal categories with corners
- Resources flow between different components of the systems through the corners

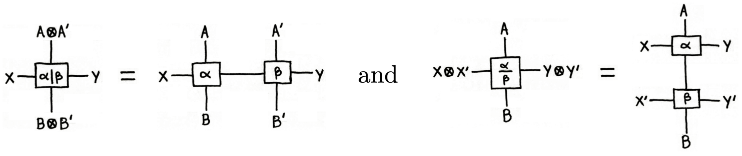
# Single object double category

Start with single object double category



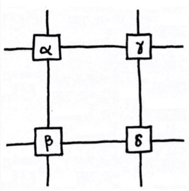

When cells $\alpha$ & $\beta$ have appropriate matching boundary types, we can have **horizontal composition** $\alpha|\beta$ or **vertical composition** $\frac{\alpha}{\beta}$ as defined below.



# Single object double category: Interchange rule

This horizontal & vertical composition needs to satisfy the interchange rule

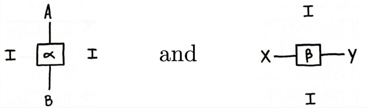
$$\frac{\alpha}{\beta}\Big|\frac{\gamma}{\delta} = \frac{\alpha|\gamma}{\beta|\delta}$$



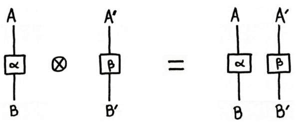
# Monoidal categories

* We shall refer to the horizontal 1-cells as **sequential** 1-cells, and vertical 1-cells as **parallel** 1-cells.

Given a single object double category $\mathbb{A}$, we get two strict monoidal categories, **S**$(\mathbb{A})$, **P**$(\mathbb{A})$ as follows:

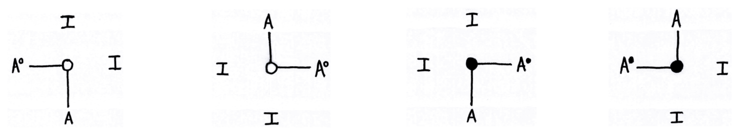



The tensor product of sequential morphisms given as follows:

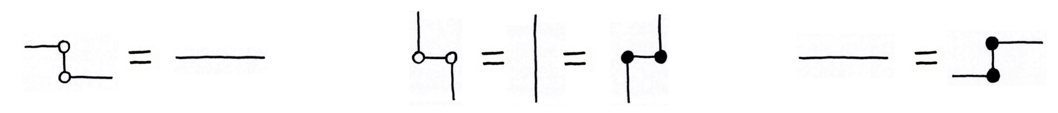


# Adding Cornerings

A single object double category is a pro-arrow equipment if for every horizontal 1-cell $A$, there exists parallel 1-cells $A^\circ$ and $A^\bullet$ with $A^\circ \neq A^\bullet$ and the following 2-cells:




called ○-corners and ●-corners, respectively, which satisfy the yanking equations:



# Resource theories as strict monoidal categories

This work considers a concurrent process as a resource theory (strict symmetric monoidal category).

A strict symmetric monoidal category can be interpreted as resource theory [1] -

**Objects:** Resources

**Arrows:** Resource transformations that can be implemented without any cost

**Composition:** Sequential composition of resource transformations

**Tensor:** Parallel composition of resources and transformations

**Unit object:** Trivial resource

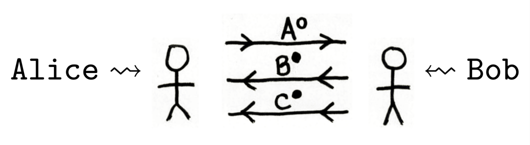
[1] Coecke, Spekkens, Fritz (2013) “A mathematical theory of resources”

# $\mathbb{A}$-valued exchanges

Given a resource theory (i.e., a strict SMC) $\mathbb{A}$, the *$\mathbb{A}$-valued exchanges* $\mathbb{A}^{\circ\bullet}$ is the free monoid on the set $(\mathrm{ob}(\mathbb{A}) \times \{\circ, \bullet\})$, whose elements are denoted by $A^\circ$ and $A^\bullet$.

The monoid is NOT commutative.

Intuitively, elements of $\mathbb{A}^{\circ\bullet}$ of describe a sequence of resources moving between participants in the exchange.



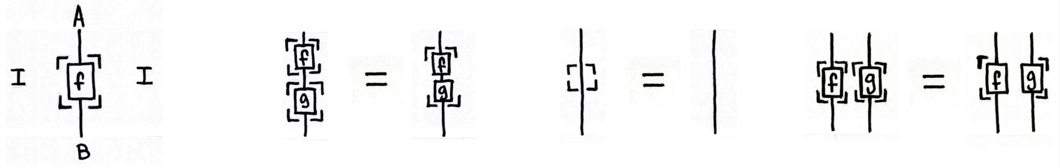
# Free cornering of $\mathbb{A}$

Given a resource theory (i.e., a strict SMC) $\mathbb{A}$, the free cornering of $\mathbb{A}$ is a proarrow equipment $\ulcorner\mathbb{A}\lrcorner$ where,

$\ulcorner\mathbb{A}\lrcorner_H$: are objects of $\mathbb{A}$ (Resources)

$\ulcorner\mathbb{A}\lrcorner_V$ are elements of $\mathbb{A}$-valued exchange monoid (Sequence of resources on the move)

Generating 2 -cells: the $\circ$-corners and $\bullet$-corners along with a vertical cell $\ulcorner f \lrcorner$ for each $f : A \to B$ subject to the equations:

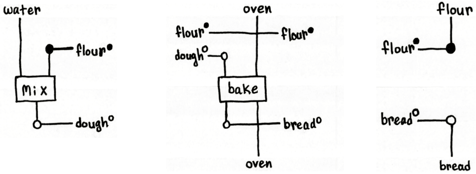


# Free cornering captures concurrency

For $\ulcorner \mathbb{A} \urcorner$, we interpret

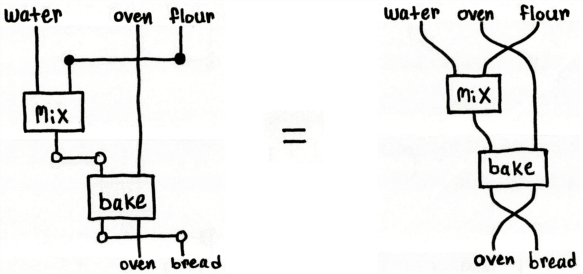
1. $\ulcorner \mathbb{A} \urcorner_{H}$ = as resources
2. $\ulcorner \mathbb{A} \urcorner_{V} = \mathbb{A}^{\circ\bullet}$ exchange of resources from one party to the other.
3. cells of $\ulcorner \mathbb{A} \urcorner$ as **concurrent transformations.** Each cell describes a way to transform the collection of resources given by the top boundary into that given by the bottom boundary via participating in $A$-exchanges along the left and right boundaries.

For example, in the free cornering of our bread category,



# $\mathsf{S}\ulcorner\mathbb{A}\urcorner$ is isomorphic to the resource theory

Combining the prior three bread concurrent transformations, we get

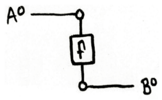



**Theorem:** *There is an isomorphism of categories* $\mathbf{S}\ulcorner\mathbb{A}\urcorner \cong \mathbb{A}$.

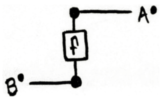
# $\mathbf{P}\ulcorner\mathbb{A}\lrcorner$ the category of resource transducers

Consider the category $\mathbf{P}\ulcorner\mathbb{A}\lrcorner$ as the category of resource transducers.

Lemma: There are strong monoidal functors $(-)^\circ : \mathbb{A} \to \mathbf{P}\ulcorner\mathbb{A}\lrcorner$ and $(-)^\bullet : \mathbb{A}^{\mathrm{op}} \to \mathbf{P}\ulcorner\mathbb{A}\lrcorner$ defined respectively on $f : A \to B$ of $\mathbb{A}$ by:




$and$



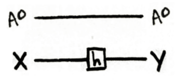
# $\mathrm{P}\ulcorner\mathbb{A}\urcorner$ is a linear actegory

For the category of resource transducers $\mathbf{P}\ulcorner\mathbb{A}\urcorner$ define -

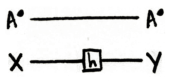
$$\circ : \mathbb{A} \times \mathbf{P}\ulcorner\mathbb{A}\urcorner \to \mathbf{P}\ulcorner\mathbb{A}\urcorner; f \circ h = f^{\circ} \otimes h$$
$$\bullet : \mathbb{A}^{\mathrm{op}} \times \mathbf{P}\ulcorner\mathbb{A}\urcorner \to \mathbf{P}\ulcorner\mathbb{A}\urcorner; f \bullet h = f^{\bullet} \otimes h$$

for all resource transducer $h \in \mathbf{P}\ulcorner\mathbb{A}\urcorner$.

Lemma In $\mathbf{H}\ulcorner\mathbb{A}\urcorner$, for each $A$, the functors $A \circ -$ is left adjoint to $A \bullet -$:

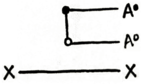



*and*

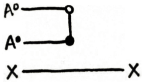


# $\mathbf{P}\ulcorner\mathbb{A}\urcorner$ is a linear actegory (cont...)

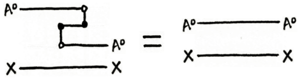
We seek families of morphisms $\eta_{A,X} : X \to A \bullet (A \circ X)$ and $\varepsilon_{A,X} : A \circ (A \bullet X) \to X$ in $\mathbf{P}\ulcorner\mathbb{A}\urcorner$ that satisfy the triangle identities. Define $\eta_{A,X}$ and $\epsilon_{A,X}$, repressively, by



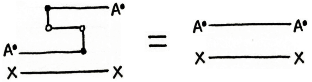

and




By the yanking equations, the triangles identities hold, as shown below.




and




**Theorem:** Let $\mathbb{A}$ be a resource theory. Then, $\mathbf{P}\ulcorner\mathbb{A}\urcorner$ is a linear actegory.

Concurrency, done!

On to quantum ...

# Categorical quantum mechanics

# Introduction

Linear logic captures the essence of quantum mechanics owing to its resource-sensitive character.

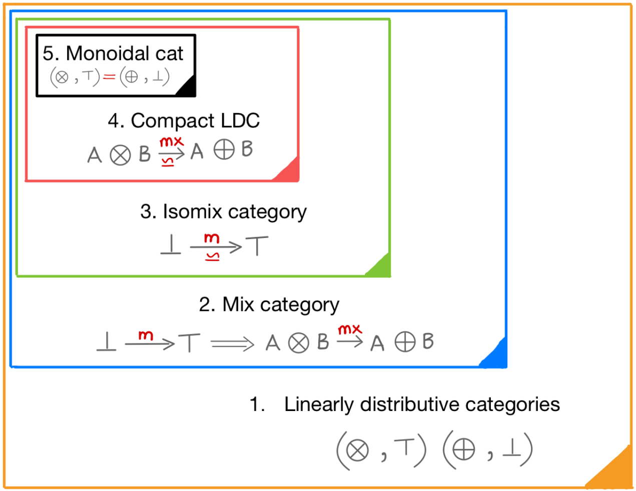
(In linear logic) *Thou shall not duplicate or discard an arbitrary resource* $\approx$ (By no-cloning theorem)*Thou shall not duplicate an arbitrary quantum state*

Categorical Quantum Mechanics (CQM) uses this connection to develop a diagrammatic framework based on the graphical calculus of monoidal categories for describing quantum mechanics

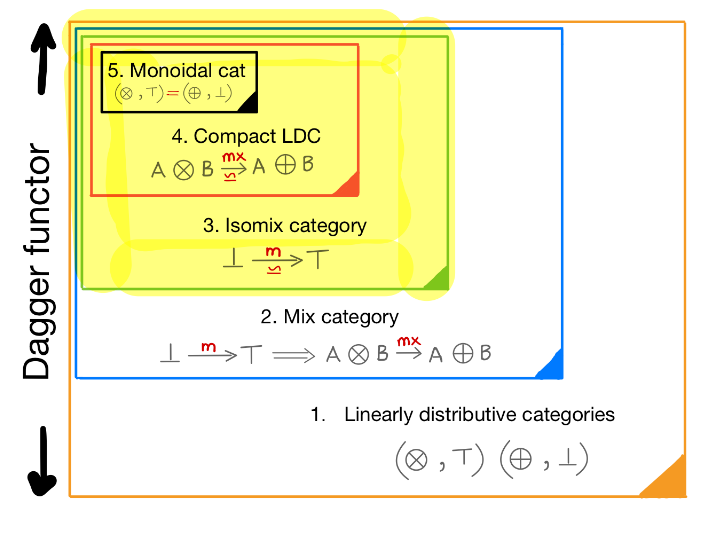
CQM introduced a dagger functor for monoidal and compact closed which abstracts unitary evolution of quantum systems.

My thesis introduced dagger isomix and mixed unitary categories as a framework for reasoning about arbitrary dimensional quantum structures.

# The LDC Rainbow



# The LDC Rainbow - Quantum

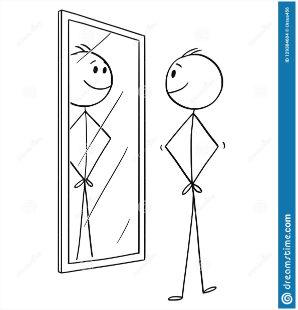


# Dagger monoidal category

In a $\dagger$-**monoidal category**, dagger is a contravariant functor:

- Stationary on objects $A = A^\dagger$
- Involution on maps $f^{\dagger\dagger} = f$
- Coherent with the tensor $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- The basic natural isomorphisms are unitary:

$$a_\otimes^{-1} = a_\otimes^\dagger; \quad u_\otimes^{-1} = u_\otimes^\dagger; \quad c_\otimes^{-1} = c_\otimes^\dagger$$



# Dagger for LDCs

The definition of $\dagger : \mathbb{X}^{op} \to \mathbb{X}$ cannot be directly imported to LDCs because the dagger minimally has to **flip the tensor products**: $(A \otimes B)^\dagger = A^\dagger \oplus B^\dagger$.



**Why?** If dagger is identity-on-objects, then the linear distributor degenerates to associator:

$$\frac{\delta^R : (A \oplus B) \otimes C \to A \oplus (B \otimes C)}{(\delta_R)^\dagger : A \oplus (B \otimes C) \to (A \oplus B) \otimes C}$$

# Dagger for LDCs

The dagger for an LDC is a contravariant Frobenius functor which is a linear involutive equivalence.

A †-LDC is a LDC $\mathbb{X}$ with a dagger functor $\dagger : \mathbb{X}^{op} \to \mathbb{X}$ and the natural isomorphisms:

$$\begin{aligned}
\text{tensor laxtors: } \lambda_\oplus &: A^\dagger \oplus B^\dagger \to (A \otimes B)^\dagger \\
\lambda_\otimes &: A^\dagger \otimes B^\dagger \to (A \oplus B)^\dagger \\
\text{unit laxtors: } \lambda_\top &: \top \to \bot^\dagger \\
\lambda_\bot &: \bot \to \top^\dagger \\
\text{involutor: } \iota &: A \to A^{\dagger\dagger}
\end{aligned}$$

such that certain coherence conditions hold.

# †-isomix categories

A †-mix category is a †-LDC with $m : \bot \to \top$ such that:

$$\begin{array}{ccc} \bot & \xrightarrow{m} & \top \\ {\scriptstyle \lambda_\bot}\downarrow & & \downarrow{\scriptstyle \lambda_\top} \\ \top^\dagger & \xrightarrow[m^\dagger]{} & \bot^\dagger \end{array}$$

**Lemma 1**: The following diagram commutes in a mix †-LDC:

$$\begin{array}{ccc} A^\dagger \otimes B^\dagger & \xrightarrow{\mathsf{mix}} & A^\dagger \oplus B^\dagger \\ {\scriptstyle \lambda_\otimes}\downarrow & & \downarrow{\scriptstyle \lambda_\oplus} \\ (A \oplus B)^\dagger & \xrightarrow[\mathsf{mix}^\dagger]{} & (A \otimes B)^\dagger \end{array}$$

For a †-mix category, if m is an isomorphism, then $\mathbb{X}$ is a †-isomix category.

# *Quantum* Message Passing Logic

Research in progress with Robin (2023)

# Dagger linear actegory

Recall that a linear actegory is loosely a monoidal category acting on an LDC. A dagger linear actegory is a dagger monoidal category acting on a dagger isomix category.

A dagger linear actegory is a $\mathbb{A}$-linear actegory ($(\mathbb{A}, *, I)$ is the monoidal category and let $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ be the LDC) such that:

- $(\mathbb{A}, *, I)$ is a $\dagger$-monoidal category

- $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is $\dagger$-isomix category

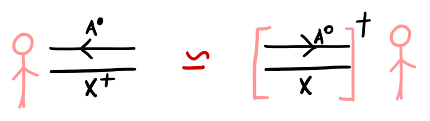
- for all $A \in \mathbb{A}$, and for all $X \in \mathbb{X}$, there exists natural isomorphisms,

$$(\phi_\bullet)_X : A \bullet X^\dagger \to (A \circ X)^\dagger$$
$$(\phi_\circ)_X : A \circ X^\dagger \to (A \bullet X)^\dagger$$

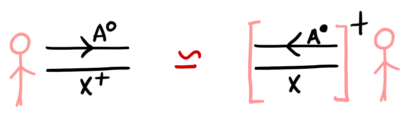
satisfying the following coherences:

# Taking a closer look at the isomorphisms:

- for all $A \in \mathbb{A}$, and for all $X \in \mathbb{X}$, there exists natural isomorphisms,




$$(\phi_\bullet)_X : A \bullet X^\dagger \xrightarrow{\simeq} (A \circ X)^\dagger$$




$$(\phi_\circ)_X : A \circ X^\dagger \xrightarrow{\simeq} (A \bullet X)^\dagger$$

Somehow, the distinction between left and right boundary seems to vanish ...

# Dagger linear actegory (cont...)

- Interaction of the nat. isos with the involutor (2 coh.)

$$
\begin{array}{ccc}
A \circ X & \xrightarrow{A \circ \iota} & A \circ X^{\dagger\dagger} \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \phi_\circ} \\
(A \circ X)^{\dagger\dagger} & \xrightarrow[\phi_\bullet^\dagger]{} & (A \bullet X^\dagger)^\dagger
\end{array}
$$

- Interaction of the nat. isos with $u_\bullet$ and $u_\circ$ (2 coh.)

$$
\begin{array}{ccc}
X^\dagger & \xrightarrow{u_\bullet} & I \bullet X^\dagger \\
 & \searrow{\scriptstyle u_\circ^\dagger} & \downarrow{\scriptstyle \phi_\bullet} \\
 & & (I \circ X)^\dagger
\end{array}
$$

# Dagger linear actegory (cont...)

- Interaction of the nat. isos with $a^*_\bullet$ and $a^*_\circ$ (2 coh.)

$$
\begin{array}{ccc}
A \bullet (B \bullet X^\dagger) & \xrightarrow{a^*_\bullet} & (A * B) \bullet X^\dagger \\
\downarrow{\scriptstyle A \bullet \phi_\bullet} & & \\
A \bullet (B \circ X)^\dagger & & \downarrow{\scriptstyle \phi_\bullet} \\
\downarrow{\scriptstyle \phi_\bullet} & & \\
(A \circ (B \circ X))^\dagger & \xrightarrow[a^{*\dagger}_\circ]{} & ((A * B) \circ X)^\dagger
\end{array}
$$

# Dagger linear actegory (cont...)

- Interaction of the nat. isos with $a^{\circ}_{\otimes}$ and $a^{\bullet}_{\oplus}$ (2 coh.)

$$\begin{array}{ccc}
(A \bullet X^\dagger) \oplus Y^\dagger & \xrightarrow{a^{\bullet}_{\oplus}} & A \bullet (X^\dagger \oplus Y^\dagger) \\
\downarrow{\scriptstyle \phi_\bullet \oplus Y^\dagger} & & \downarrow{\scriptstyle 1 \bullet \lambda_\oplus} \\
(A \circ X)^\dagger \oplus Y^\dagger & & A \bullet (X \otimes Y)^\dagger \\
\downarrow{\scriptstyle \lambda_\oplus} & & \downarrow{\scriptstyle \phi_\bullet} \\
((A \circ X) \otimes Y)^\dagger & \xrightarrow[a^{\circ\dagger}_{\otimes}]{} & (A \circ (X \otimes Y))^\dagger
\end{array}$$

# Dagger linear actegories (cont...)

Interaction with the nat. trans. $d_\otimes^\bullet$ and $d_\oplus^\circ$ (2 coh.):

$$
\begin{array}{ccc}
(A \bullet X^\dagger) \otimes Y^\dagger & \xrightarrow{d_\otimes^\bullet} & A \bullet (X^\dagger \otimes Y^\dagger) \\
\Big\downarrow{\scriptstyle \phi_\bullet \otimes 1} & & \Big\downarrow{\scriptstyle 1 \bullet \lambda_\otimes} \\
(A \circ X)^\dagger \otimes Y^\dagger & & A \bullet (X \oplus Y)^\dagger \\
\Big\downarrow{\scriptstyle \lambda_\otimes} & & \Big\downarrow{\scriptstyle \phi_\bullet} \\
((A \circ X) \oplus Y)^\dagger & \xrightarrow[d_\oplus^{\circ\dagger}]{} & (A \circ (X \oplus Y))^\dagger
\end{array}
$$

# Dagger linear actegories (cont...)

Interaction with the nat. trans. $d_{\bullet}^{\circ}$ (1 coh.):

$$
\begin{array}{ccc}
B \circ (A \bullet X^{\dagger}) & \xrightarrow{d_{A \bullet -}^{B \circ -}} & A \bullet (B \circ X^{\dagger}) \\
{\scriptstyle 1 \circ \phi_{\bullet}} \downarrow & & \downarrow {\scriptstyle 1 \bullet \phi_{\circ}} \\
B \circ (A \circ X)^{\dagger} & & A \bullet (B \bullet X)^{\dagger} \\
{\scriptstyle \phi_{\circ}} \downarrow & & \downarrow {\scriptstyle \phi_{\bullet}} \\
(B \bullet (A \circ X))^{\dagger} & \xrightarrow[(d_{B \bullet -}^{A \circ -})^{\dagger}]{} & (A \circ (X \oplus Y))^{\dagger}
\end{array}
$$

# Example

**Example of a dagger linear actegory:** Mixed unitary categories

What is a mixed unitary category?

## The core of a mix category

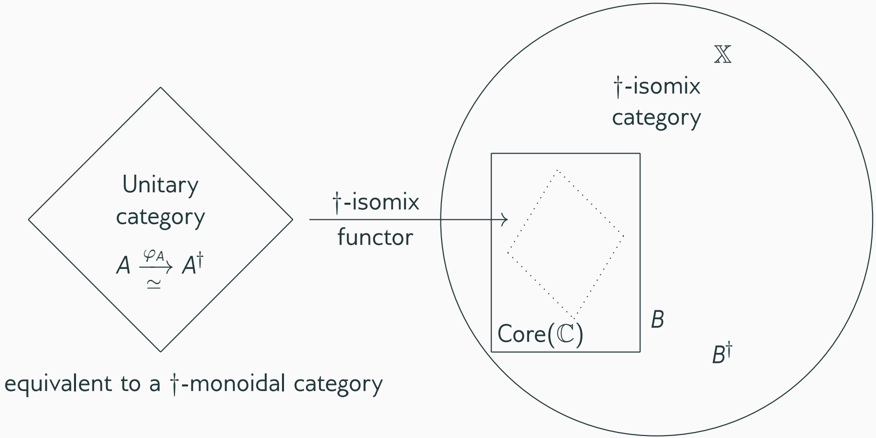
The core of a mix category, $\mathrm{Core}(\mathbb{X}) \subseteq \mathbb{X}$, is the full subcategory determined by objects $U \in \mathbb{X}$ for which the natural transformation is also an isomorphism:

$$U \otimes A \xrightarrow{\mathsf{mx}_{U,A}} U \oplus A$$

The core of a **mix** category is closed to $\otimes$ and $\oplus$.

The core of an isomix category contains the monoidal units $\top$ and $\bot$.

# Mixed unitary categories (MUCs)




A mixed unitary category, $M : \mathbb{U} \to \mathbb{C}$, is

†-isomix functor: unitary category $\to$ †-isomix category

# A MUC is a linear actegory

Theorem: A mixed unitary category with unitary duals is a linear actegory.

Proof (Sketch):

Let $M : \mathbb{U} \to \mathbb{C}$ be a mixed unitary category with unitary duals.

For all $U \in \mathbb{U}$,

$$U \circ C := M(U) \otimes C$$

$$U \bullet C := M(U^\dagger) \oplus C$$

We need $\mathbb{U}$ to have dagger duals ($U \dashv U^\dagger$) to get the adjunction $U \circ - \dashv U \bullet -$

## A MUC is a linear actegory (cont...)

for all $U \in \mathbb{U}$, we need a family of maps:

$$\eta_X : X \to U \bullet (U \circ X) :=?$$

$$\epsilon_X : U \circ (U \bullet X) \to X :=?$$

And have to define the six natural isomorphisms and the three natural transformations:

....

# A MUC is a dagger linear actegory (cont...)

Theorem: A mixed unitary category with unitary duals is a dagger linear actegory.

for all $A \in \mathbb{A}$ what are the following family of maps?

$$(\phi_\bullet)_X : A \bullet X^\dagger \to (A \circ X)^\dagger \ := \ ?$$
$$(\phi_\circ)_X : A \circ X^\dagger \to (A \bullet X)^\dagger \ := \ ?$$

# Examples of MUCs

- Every $\dagger$-monoidal category is a MUC
- FinRel $\hookrightarrow$ FRel: Finite relations embedded into finiteness relations
- Mat($\mathbb{C}$) $\hookrightarrow$ FMat($\mathbb{C}$): Complex finite dimensional matrices embedded into finiteness matrices over a commutative rig $R$
- FHilb $\hookrightarrow$ $\mathsf{Chus}_I(\mathsf{Vec}(\mathbb{C}))$: Finite-dimensional Hilbert spaces embedded into Chu spaces over complex vector spaces
- Unitary construction: Given any $\dagger$-isomix category $\mathbb{C}$ one can construct a canonical MUC, Unitary($\mathbb{C}$) $\hookrightarrow$ $\mathbb{C}$, by choosing its pre-unitary objects.

Unitary($\mathbb{C}$):

**Objects:** Pre-unitary objects $(U, \alpha)$;

**Maps:** $(U, \alpha) \xrightarrow{f} (V, \beta)$ where $U \xrightarrow{f} V$ is any map of $\mathbb{X}$.

# Speculation

How does the definition of dagger linear actegory fit in Chad Nester's framework?

**Read with caution:** Hence, what is a 'dagger proarrow equipment' for a single object double category or what is a dagger double category?

Chad notices, for any resource theory $\mathbb{A}$, there exists an contravariant involution

$$(-)^* : \mathbf{P}\ulcorner\mathbb{A}\urcorner^{\mathsf{op}} \to \mathbf{P}\ulcorner\mathbb{A}\urcorner$$

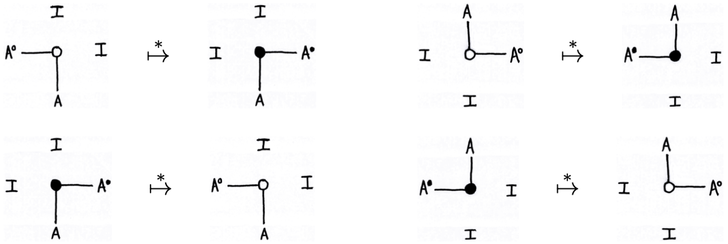
given as follows:

$$(A^\circ)^* = A^\bullet \qquad (A^\bullet)^* = A^\circ$$

# Speculation (cont...)

The involution on $\mathbf{P}\ulcorner\mathbb{A}\urcorner$ allows an contravariant involution $(-)^*$ to be defined on $\ulcorner\mathbb{A}\urcorner$:

$\ulcorner f \urcorner^* = \ulcorner f \urcorner$ along with:



Can we similarly define a dagger involution?

# For future

* Build a toy model of dagger linear actegories by extending Chad Nester's framework?

* Describe quantum communication protocols

* Term calculus of quantum message passing logic

* Proof theory of quantum message passing logic

* A curry-Howard Lambek like correspondence

* Programming syntax for this logic

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