Double Grothendieck:

Double Fibrations and Double Colimits

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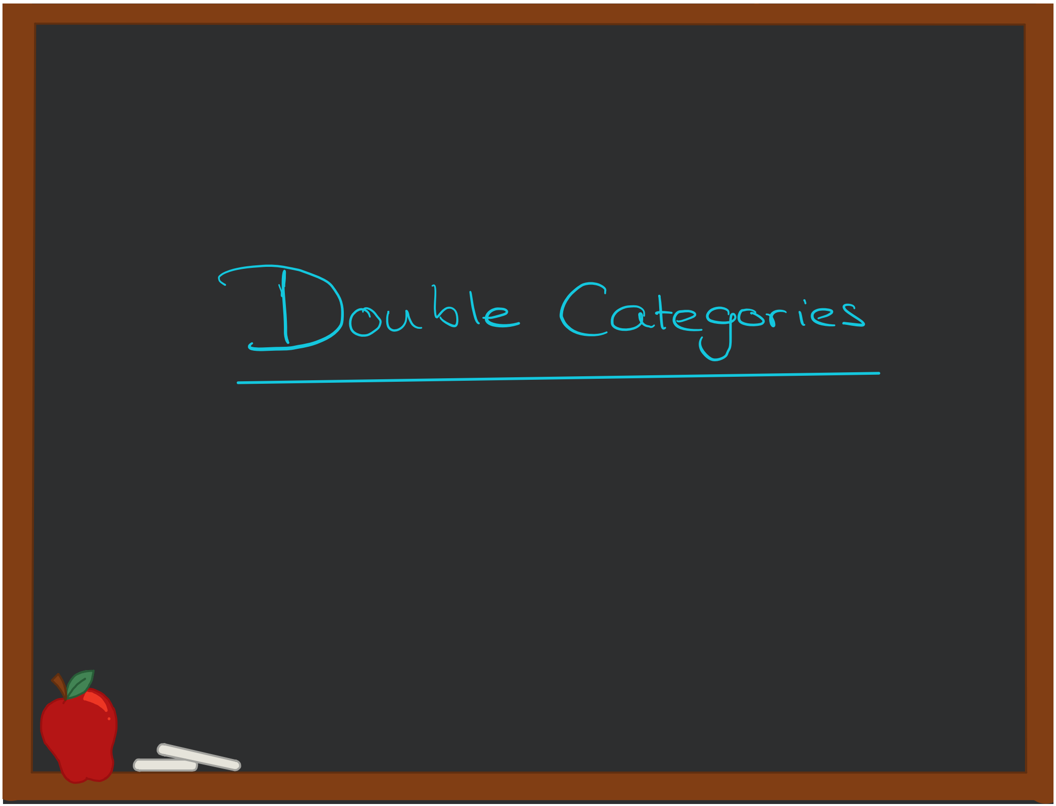

Double Categories

Pseudo category objects

Definition For a 2-category $\mathcal{K}$, a pseudo category $\mathbb{C}$ in $\mathcal{K}$ consists of a diagram:

$$
\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "\otimes"] & C_1 \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & C_0 \arrow[l, "y"']
\end{tikzcd}
$$

and iso. 2-cells:

$$
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\otimes"'] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "{(y,1)}"] \arrow[dr, bend right, "1_{C_1}"'] \arrow[dr, phantom, "\overset{l}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] & C_1 \arrow[l, "{(1,y)}"'] \arrow[dl, bend left, "1_{C_1}"] \arrow[dl, phantom, "\overset{r}{\cong}"] \\
& C_1 &
\end{tikzcd}
$$

normalized: l and r id's.

Double Categories

Definition A double category is a pseudo category object in $\underline{\mathcal{C}at}$:

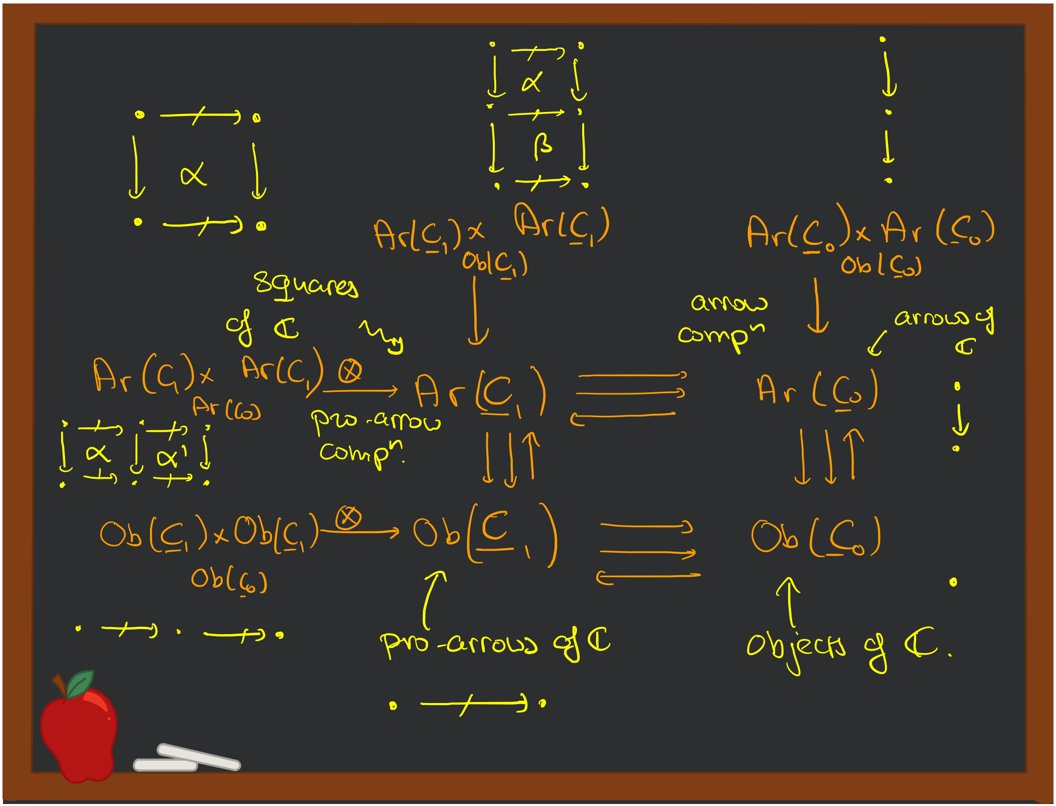
$$
\begin{tikzcd}
\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \arrow[r, "\otimes"] & \underline{C}_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & \underline{C}_0 \arrow[l, shift left=5, "y"]
\end{tikzcd}
$$

$\otimes$ ps. assoc. and ps. unitary

$\underline{C}_0$, $\underline{C}_1$ categories.

Let's spell that out :

$$
\begin{tikzcd}
 & \mathrm{Ar}(\underline{C})_1 \times_{\mathrm{Ob}(\underline{C})_1} \mathrm{Ar}(\underline{C})_1 \arrow[d] & \mathrm{Ar}(\underline{C}_0) \times_{\mathrm{Ob}(\underline{C}_0)} \mathrm{Ar}(\underline{C}_0) \arrow[d] \\
\mathrm{Ar}(\underline{C}_1) \times_{\mathrm{Ar}(\underline{C}_0)} \mathrm{Ar}(\underline{C}_1) \arrow[r, "\otimes"] & \mathrm{Ar}(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[d, shift right=2] \arrow[d] & \mathrm{Ar}(\underline{C}_0) \arrow[l, shift left=2] \arrow[d, shift right=2] \arrow[d] \\
\mathrm{Ob}(\underline{C}_1) \times_{\mathrm{Ob}(\underline{C}_0)} \mathrm{Ob}(\underline{C}_1) \arrow[r, "\otimes"] & \mathrm{Ob}(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[u, shift right=2] & \mathrm{Ob}(\underline{C}_0) \arrow[l, shift left=2] \arrow[u, shift right=2]
\end{tikzcd}
$$



α
α
β
Ar(C1) × Ob(C1) Ar(C1)
Ar(C0) × Ob(C0) Ar(C0)
squares of C
arrow compn
arrows of C
Ar(C1) × Ar(C0) Ar(C1)
⊗
pro-arrow compn
Ar(C1)
Ar(C0)
α α'
Ob(C1) × Ob(C0) Ob(C1)
⊗
Ob(C1)
Ob(C0)
pro-arrows of C
objects of C.

Proarrow composition is only required to be unitary and associative up to isomorphism:

there are double cells:

unitors:

$$\begin{tikzcd}
A \arrow[r, "y_B h", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "\lambda_h"] & B \arrow[d, "1_B"] \\
A \arrow[r, "h"', "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "h y_A", "\shortmid"{marking}] \arrow[d, "1_A"'] \arrow[dr, phantom, "\rho_h"] & B \arrow[d, "1_B"] \\
A \arrow[r, "h"', "\shortmid"{marking}] & B
\end{tikzcd}$$

associators:

$$\begin{tikzcd}
A \arrow[r, "gf", "\shortmid"{marking}] \arrow[d, "1_A"'] & C \arrow[r, "h", "\shortmid"{marking}] \arrow[d, phantom, "\alpha_{hgf}"] & D \arrow[d, "1_D"] \\
A \arrow[r, "f"', "\shortmid"{marking}] & B \arrow[r, "hg"', "\shortmid"{marking}] & D
\end{tikzcd}$$

Vertically invertible and satisfying coherence.

Examples

1. **Rel** : objects are sets

proarrows are relations

arrows are functions

squares :

$$\begin{tikzcd} S \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow"] & T \arrow[d, "v"] \\ S' \arrow[r, "R'"', "\shortmid"{marking}] & T' \end{tikzcd}$$

$R \subseteq S \times T$

$R' \subseteq S' \times T'$

such that

for $(s,t) \in R$, $(u(s), v(t)) \in R'$

2. $\mathrm{Span}(\mathcal{C})$, where $\mathcal{C}$ is a category with pullbacks:

obj: are objects in $\mathcal{C}$

arrows are arrows in $\mathcal{C}$.

proarrows are spans

$$S \xleftarrow{s} A \xrightarrow{t} T$$

squares: commutative diagrams

$$\begin{tikzcd}
S \arrow[d, "u"'] & A \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "w"] & T \arrow[d, "v"] \\
S' & A' \arrow[l, "s'"'] \arrow[r, "t'"] & T'
\end{tikzcd}$$

3. For $\mathcal{C}_0 = \mathbf{1}$, a double category $\mathcal{C}_1 \times \mathcal{C}_1 \xrightarrow{\otimes} \mathcal{C}_1 \xleftarrow{y} \mathbf{1}$ is just a monoidal category.

4. A 2-category $\mathcal{A}$ gives rise to double categories

- $\mathbb{V}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "v"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[d, bend right, "u"'] \arrow[d, bend left, "v"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ B \end{tikzcd}$$
in $\mathcal{A}$.

- $\mathbb{H}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$
in $\mathcal{A}$.

- $\mathbb{Q}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[r, bend left, "vh"] \arrow[r, bend right, "ku"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$
in $\mathcal{A}$.

Pseudo Double Functors

For: $\mathbb{C}: \underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \xrightarrow{\otimes} \underline{C}_1$ with $s, t: \underline{C}_1 \to \underline{C}_0$ and $y: \underline{C}_0 \to \underline{C}_1$

and $\mathbb{D}: \underline{D}_1 \times_{\underline{D}_0} \underline{D}_1 \xrightarrow{\otimes} \underline{D}_1$ with $s, t: \underline{D}_1 \to \underline{D}_0$ and $y: \underline{D}_0 \to \underline{D}_1$

a pseudo functor consists of

$$F_0: \underline{C}_0 \longrightarrow \underline{D}_0 \quad \text{and} \quad F_1: \underline{C}_1 \longrightarrow \underline{D}_1$$

with comparison cells:

$$\begin{tikzcd}
\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\varphi_1}{\underset{\cong}{\Rightarrow}}"] & \underline{C}_1 \arrow[d, "F_1"] \\
\underline{D}_1 \times_{\underline{D}_0} \underline{D}_1 \arrow[r, "\otimes"'] & \underline{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C}_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{z}{\underset{\cong}{\Rightarrow}}"] & \underline{C}_1 \arrow[d, "F_1"] \\
\underline{D}_0 \arrow[r, "y"'] & \underline{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "F_1"'] \arrow[dr, phantom, "="] & \underline{C}_0 \arrow[d, "F_0"] \\
\underline{D}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \underline{D}_0
\end{tikzcd}$$

Stricter!

The Grothendieck Construction

Recall a classical result from SGA 4:

- For an indexing pseudo functor $F : \mathcal{A}^{op} \longrightarrow \underline{Cat}$ with structure isomorphisms

$$\varphi_A : 1_{FA} \overset{\sim}{\Rightarrow} F(1_A)$$

$$\varphi_{g,f} : Fg \circ Ff \overset{\sim}{\Rightarrow} F(g \circ f)$$

the category of elements $\int_{\mathcal{A}} F \xrightarrow{\pi_F} \mathcal{A}$ has

<u>objects</u>: (A, x), $A \in Obj(\mathcal{A})$, $x \in Obj(F(A))$

<u>arrows</u>: $(g, \psi) : (A, x) \longrightarrow (B, y)$ with $g : A \to B$ and $\psi : x \longrightarrow F(g)(y)$ in $F(A)$.

identities: $\mathrm{id}_{(A,x)} = (\mathrm{id}_A, (\varphi_A)_x)$

$$(\varphi_A)_x : x \to F(1_A)(x)$$

composition:

for $(A,x) \xrightarrow{(g_1,\psi_1)} (B,y) \xrightarrow{(g_2,\psi_2)} (C,z)$:

$$(g_2,\psi_2)\circ(g_1,\psi_1) = (g_2\circ g_1,\ \varphi_{g_1,g_2}\circ F(g_1)(\psi_2)\circ\psi_1)$$

$$x \xrightarrow{\psi_1} F(g_1)(y) \xrightarrow{F(g_1)(\psi_2)} F(g_1)F(g_2)(z) \xrightarrow{\varphi_{g_1,g_2}} F(g_1 g_2)(z)$$

Properties of $\int_{\mathcal{A}} F \xrightarrow{\pi_F} \mathcal{A}$.

- $\pi_F : \int_{\mathcal{A}} F \longrightarrow \mathcal{A}$ is defined by:

$$(A, x) \longmapsto A$$
$$(g, \varphi) \longmapsto g$$

- π_F is a <u>fibration</u>

Fibrations

Fibrations – Cartesian Arrows

Let $p : \underline{E} \longrightarrow \underline{B}$ be a functor. An arrow $\gamma : \cancel{e \to e'}$ $e' \to e$ in $\underline{E}$ is Cartesian if for all

$$\begin{tikzcd}
e'' \arrow[d, dashed, "\exists!\, h"'] \arrow[dr, "\psi"] & \\
e' \arrow[r, "\gamma"'] \arrow[ur, phantom, "="] & e
\end{tikzcd} \qquad \text{in } \underline{E}$$

$$\begin{tikzcd}
pe'' \arrow[d, "p(h) = g"'] \arrow[dr, "p\psi"] & \\
pe' \arrow[r, "p\gamma"'] \arrow[ur, phantom, "="] & pe
\end{tikzcd} \qquad \text{in } \underline{B}$$

there is a unique lifting h.

Fibrations

① $p : \underline{E} \longrightarrow \underline{B}$ is a fibration if for any

$$e' \xrightarrow{\varphi} e \qquad \text{in } \underline{E}$$

$$b \xrightarrow[f]{} pe \qquad \text{in } \underline{B}$$

there is a Cartesian lifting φ (of f to e)

② A cleavage for p consists of a choice of a Cartesian lifting for each f and e.

π_F is a fibration

$F : \mathcal{A}^{op} \to \underline{Cat}$

$$\begin{tikzcd} \int_{\mathcal{A}} F \arrow[d, "\pi_F"'] \\ \mathcal{A} \end{tikzcd}$$

$(B, Ff(x)) \longrightarrow (A, x)$

$(f,$

$x \in FA$

$\in FB$

$B \xrightarrow{f} A$

π_F is a fibration

$$\begin{tikzcd} \int_{\mathcal{A}} F \arrow[d, "\pi_F"'] & (B, Ff(x)) \arrow[r, "{(f,\, 1_{Ff(x)})}"'] & (A, x) \\ \mathcal{A} & B \arrow[r, "f"'] & A \end{tikzcd}$$

Canonical Cartesian morphisms:

$$(f,\ 1_{Ff(x)})$$

they form a cleavage.

Result: for a fixed base category $\mathbf{B}$, the Grothendieck construction extends to a 2-equivalence:

$$\int_{B} : [\mathbf{B}^{op}, \mathbf{Cat}] \xrightarrow{\sim} \mathbf{Fib}(\mathbf{B})$$

$[\mathbf{B}^{op}, \mathbf{Cat}]$: pseudo-functors, pseudo natural transformations, modifications

$\mathbf{Fib}(\mathbf{B})$: fibrations over $\mathbf{B}$, cartesian functors, vertical natural transformations

$$\begin{tikzcd} \mathbf{E} \arrow[rr, bend left, "\Phi"] \arrow[rr, bend right, "\Psi"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "p"'] & & \mathbf{F} \arrow[dl, "q"] \\ & \mathbf{B} & \end{tikzcd} \qquad q(\alpha_e) = 1_{p(e)} .$$

Fibrations over an arbitrary Base

$\underline{\mathrm{Fib}}$ is the 2-category:

objects: cloven fibrations $P : \underline{E} \to \underline{B}$ (arbitrary $\underline{B}$)

arrows: $f : (P : \underline{E} \to \underline{B}) \longrightarrow (P' : \underline{E}' \to \underline{B}')$

is a commutative square:

$$\begin{tikzcd} \underline{E} \arrow[r, "f^\top"] \arrow[d, "P"'] & \underline{E}' \arrow[d, "P'"] \\ \underline{B} \arrow[r, "f^\perp"'] & \underline{B}' \end{tikzcd}$$

, $f^\top$ preserves Cartesian arrows.

2-cells: $\alpha : f \Rightarrow g$:

$$\begin{tikzcd} \underline{E} \arrow[r, bend left, "f^\top"] \arrow[r, bend right, "g^\top"'] \arrow[r, phantom, "\Downarrow\alpha^\top"] \arrow[d, "P"'] & \underline{E}' \arrow[d, "P'"] \\ \underline{B} \arrow[r, bend left, "f_\perp"] \arrow[r, bend right, "g_\perp"'] \arrow[r, phantom, "\Downarrow\alpha_\perp"] & \underline{B}' \end{tikzcd}$$

Commutative cylinder.

$\underline{\mathrm{cFib}} \subseteq \underline{\mathrm{Fib}}$ is the locally full sub-2-category where we require the arrows $f^{\top}$ to preserve the cleavages.

We obtain two equivalences of 2-categories:

- $\underline{\mathrm{Fib}} \simeq \underline{\mathrm{ICat}}$

$$
\begin{tikzcd}
\underline{B} \arrow[d] \arrow[dr, phantom, "\Downarrow"] \arrow[r] & \underline{\mathrm{Cat}} \\
\underline{B'} \arrow[ur] &
\end{tikzcd}
$$

pseudo natural transf^{ns}

- $\underline{\mathrm{cFib}} \simeq \underline{\mathrm{ICat}}_s$

strict ntl. transformations

Examples of Fibrations

- $\underline{\mathrm{Mod}} \longrightarrow \underline{\mathrm{Ring}}$ is a fibration (and an opfibration)

$(R, M) \longmapsto R$

M a left R-module

$(R_1, f^*M) \qquad (R_2, M)$

$$R_1 \xrightarrow[f]{} R_2$$

f^*M : restriction of scalars

- <u>Codomain</u> <u>fibration</u> over a category $\underline{C}$ with pullbacks:

$\mathrm{Arr}_s(\underline{C}) = \underline{C}^2$ has obj. $\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array}$ in $\underline{C}$

and arrows:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "k"] & Y' \end{tikzcd}$$

comm. in $\underline{C}$.

The codomain functor $\underline{C}^2 \longrightarrow \underline{C}$ is a fibration and opfibration; cartesian arrows are pullback squares.

Examples (Continued)

- for $\underline{C}$ any category, $\mathrm{Fam}(\underline{C})$ is the category of set-indexed families of obj.'s in $\underline{C}$:

$$(C_i)_{i \in I}$$

morphisms:

$$(C_i)_{i\in I} \xrightarrow[(f, (\phi_i)_{i\in I})]{} (D_j)_{j \in J}$$

$f : I \to J$ function

$\phi_i : C_i \to D_{f(i)}$

This is the Grothendieck construction for

$$\underline{\mathrm{Set}}^{op} \longrightarrow \underline{\mathrm{Cat}}$$

$$I \longmapsto \prod_{i\in I} \underline{C}$$

The Category of Elements is also an oplax colimit for F as diagram in $\underline{\mathbf{Cat}}$:

- We have a universal cone:

$$
\begin{tikzcd}
A \arrow[dd, "g"'] & FA \arrow[dr, "\varepsilon_A"] \arrow[d, phantom, "\quad\;\; \Downarrow \varepsilon_g"] & \\
 & {} & \int_{\mathcal{A}} F \\
B & FB \arrow[uu, "Fg"] \arrow[ur, "\varepsilon_B"'] &
\end{tikzcd}
$$

$$\varepsilon_A : \quad x \longmapsto (A, x)$$

$$(\psi : x \to y) \longmapsto \left[(A, x) \xrightarrow[(1_A,\ \varphi_A \circ \psi)]{} (A, y) \right]$$

$$(\varepsilon_g)_y = (g, 1_{Fg(y)}) : (A, Fg(y)) \longrightarrow (B, y)$$