# FMCS Tutorial

on Double Fibrations

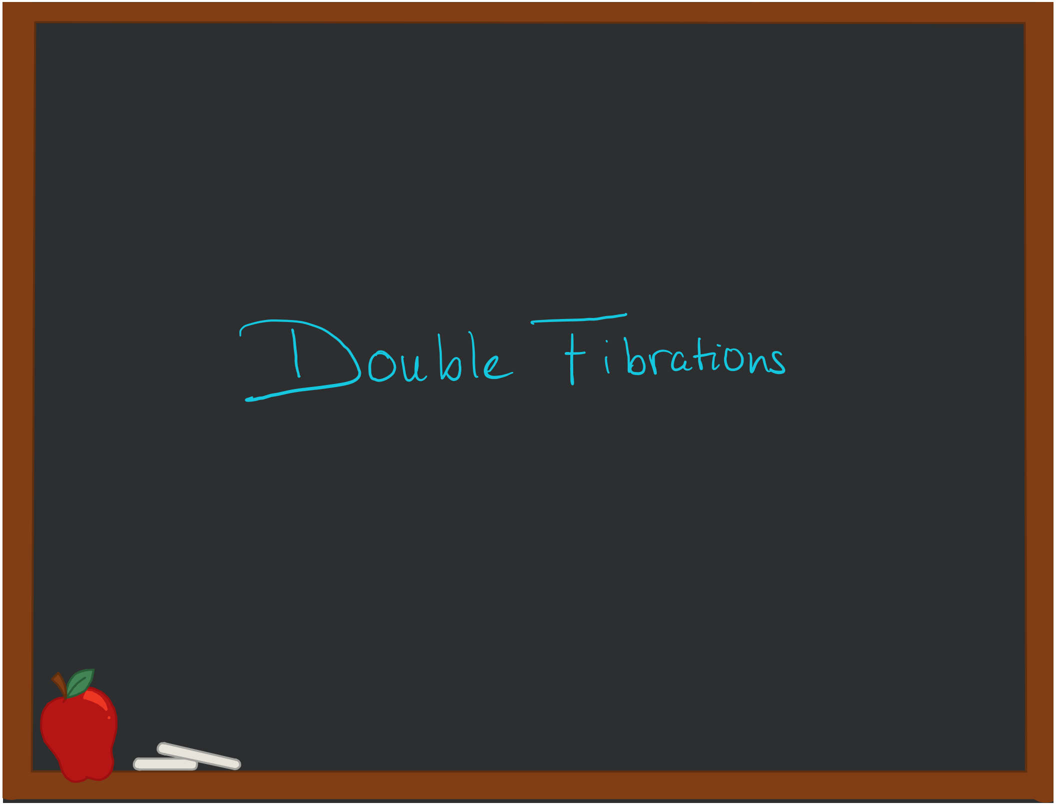
Double Grothendieck

Double Colimits

Part II

## Goal: Introduce a Double Category of Elements

- with a corresponding notion of double fibration
- extending the monoidal Grothendieck construction [Moeller, Vasilakopoulou, TAC 2020]
- Jaz-Myers' double Grothendieck constructions for open dynamical systems as special cases [EPTC 2021]
- structured and decorated cospans as special cases. [Baez-Courser-Vasilakopoulou, 2020, 2022] [Patterson, 2023]
- extending the discrete case [Lambert, TAC 2021]


Double Fibrations

Since it is not obvious what the codomain of an indexing functor

$$\mathbb{A} \longrightarrow ?? \qquad (\mathbb{A} \text{ dbl cat}^{y})$$

should be, we start with double fibrations.

# Double Fibrations

First Idea: a double category is a pseudo cat$^{y}$ in Cat $\rightsquigarrow$ a double fibration is a pseudo cat$^{y}$ in Fib.

This would amount to:

$$
\begin{tikzcd}
E_1 \times_{E_0} E_1 \arrow[r, "\otimes^{\top}"] \arrow[d, "p_1 \times_{p_0} p_1"] & E_1 \arrow[r, shift left=3, "s^{\top}"] \arrow[r, shift right=3, "t^{\top}"'] \arrow[d, "p_1"] & E_0 \arrow[l, "y^{\top}"] \arrow[d, "p_0"] \\
B_1 \times_{B_0} B_1 \arrow[r, "\otimes^{\perp}"] & B_1 \arrow[r, shift left=3, "s^{\perp}"] \arrow[r, shift right=3, "t^{\perp}"'] & B_0 \arrow[l, "y^{\perp}"]
\end{tikzcd}
$$

... strict dbl functor with extra properties?

# Double fibrations

**Problems:**

- Fib doesn't have all 2-pullbacks required for this.
- We require the same fibrational strictness for $s$ and $t$ that we require for $y$ and $\otimes$.

**Solution:** We will require that $s$ and $t$ are in cFib. (i.e., they preserve cleavages)

# Double Fibrations

<u>Definition:</u> A <u>double fibration</u> is a (strict) double functor $P: \mathbb{E} \to \mathbb{B}$ between pseudo double categories:

$$\begin{tikzcd}
\underline{E}_1 \times_{\underline{E}_0} \underline{E}_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \underline{E}_1 \arrow[r, shift left=2, "s_E"] \arrow[r, shift right=2, "t_E"'] \arrow[d, "P_1"] & \underline{E}_0 \arrow[l] \arrow[d, "P_0"] \\
\underline{B}_1 \times_{\underline{B}_0} \underline{B}_1 \arrow[r, "\otimes_B"] & \underline{B}_1 \arrow[r, shift left=2, "s_B"] \arrow[r, shift right=2, "t_B"'] & \underline{B}_0 \arrow[l]
\end{tikzcd}$$

s.t.

- $P_0$ and $P_1$ are fibrations with cleavages
- $s_E$ and $t_E$ are cleavage preserving
- $y_E$ and $\otimes_E$ are cartesian-morphism preserving.

# Examples

1. $\mathrm{Im} : \mathbb{S}\mathrm{pan}(\underline{\mathrm{Set}}) \longrightarrow \mathbb{R}\mathrm{el}$

$$\left(A \xleftarrow{s} S \xrightarrow{t} B\right) \longmapsto \left(\mathrm{Im}((s,t)) \subseteq A \times B\right)$$

is a double opfibration.

2. When $\underline{E}_0 = \underline{B}_0 = 1$, we get :

$$\begin{tikzcd}
\underline{E}_1 \times_{\underline{E}_0} \underline{E}_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \underline{E}_1 \arrow[d, "P_1"'] & 1 \arrow[l, "y"'] \arrow[d, "id"] \\
\underline{B}_1 \times_{\underline{B}_0} \underline{B}_1 \arrow[r, "\otimes_B"] & \underline{B}_1 & 1 \arrow[l, "y"']
\end{tikzcd}$$

a monoidal fibration!

3. The Grothendieck constructions given in David Jaz Myers' work are also double fibrations.

# Examples

4. For any 2-functor $P: \underline{E} \longrightarrow \underline{B}$, $P$ is a 2-fibration as in Buckley's work if and only if $\mathfrak{Q}(P): \mathfrak{Q}(\underline{E}) \longrightarrow \mathfrak{Q}(\underline{B})$ is a double fibration.

5. if $P_0$ and $P_1$ are discrete fibrations, we recover discrete double fibrations.

6. For $\mathbb{D}$ a double category, let $\mathbb{D}^2 = \left( \begin{array}{c} \mathbb{D}_1^2 \\ \downarrow\downarrow \\ \mathbb{D}_0^2 \end{array} \right)$

$$\mathrm{dom}: \mathbb{D}^2 \longrightarrow \mathbb{D}$$

is a double fibration

7. The codomain fibration extends to a double codomain fibration $\mathrm{cod}: \mathbb{D}^2 \to \mathbb{D}$ if:

- $\mathbb{D}_1$ and $\mathbb{D}_0$ have chosen finite limits
- these limits are preserved on the nose by $s$ and $t$
- and up to iso by $y$ and $\otimes$.

8. for $\mathcal{C}$ a small cat$^y$, $\mathbb{F}\mathrm{am}(\mathcal{C})$ has:

obj: $(I, \{C_i\}_{i\in I})$, or $f : I \to \mathcal{C}$

arrows:
$$\frac{(h,\alpha) : f \to g}{h : I \to J,\ \alpha : f \Rightarrow gh}$$

$$\begin{tikzcd} I \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\Rightarrow}{\alpha}"] & J \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

proarrows: natural transformations

$$\begin{tikzcd} S \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & \mathcal{C} \end{tikzcd}$$

for spans of sets

$$X \xleftarrow{d_0} S \xrightarrow{d_1} Y$$

cells: a cell from

$$\begin{tikzcd} S \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & \mathcal{C} \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} T \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\delta}{\Rightarrow}"] & W \arrow[d, "k"] \\ Z \arrow[r, "h"'] & \mathcal{C} \end{tikzcd}$$

are given by :

a morphism of spans

$$\begin{tikzcd}
X \arrow[d, "p"'] & S \arrow[l, "d_0"'] \arrow[r, "d_1"] \arrow[d, "g"'] & Y \arrow[d, "r"] \\
Z & T \arrow[l, "d_0"] \arrow[r, "d_1"'] & W
\end{tikzcd}$$

with 2-cells :

$$\begin{tikzcd}
& X \arrow[dl, "p"'] \arrow[dd, "f"] \arrow[ddl, phantom, "\overset{\alpha}{\Leftarrow}"] \\
Z \arrow[dr, "h"'] & \\
& e
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[dl, "r"'] \arrow[dd, "g"] \arrow[ddl, phantom, "\overset{\beta}{\Leftarrow}"] \\
W \arrow[dr, "k"'] & \\
& e
\end{tikzcd}$$

s.t. $(\beta * d_1)\,\theta = (\delta * g)(\alpha * d_0)$.

$$\Pi_0 : \mathbb{F}\mathrm{am}(\mathcal{C})_0 \longrightarrow \underline{\mathrm{Set}}$$

is a split fibration.

We extend this to

$$\Pi : \mathbb{F}\mathrm{am}(\mathcal{C}) \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Set}}).$$

(send proarrows to their underlying spans and cells to span morphisms)

<u>Claim:</u> this is a split double fibration.

Relation with

Street's Internal Fibrations

(a look-off trail we will not take)

# Internal Fibrations

- Internal fibrations in a 2-category were introduced by Street in 1974.
- We will use the following three 2-cat's:

$$\underline{\mathrm{Dbl}}_s \subseteq \underline{\mathrm{Dbl}} \subseteq \underline{\mathrm{Dbl}}_\ell$$

strict dbl functors — pseudo dbl functors — lax dbl functors

- In all 3 cases we use vertical transformations $\alpha : F \Rightarrow G : \mathbb{D} \rightrightarrows \mathbb{E}$:
  - $*$ for each $x$ in $\mathbb{D}$: an arrow $\alpha_x : Fx \to Gx$ in $\mathbb{E}$
  - $*$ for each pro arrow $x \overset{m}{\nrightarrow} y$ a dbl cell $\alpha_m : Fm \to Gm$ in $\mathbb{E}_1$. (natural and functorial)

## Theorem (Cruttwell, Lambert, P, Szyld)

A strict double functor is an internal fibration in DblCat if and only if it is a double fibration.

In addition, a pseudo double functor $P$

- is an internal fibration in $\mathrm{DblCat}_\ell$ iff $P_0$ and $P_1$ admit cleavages that are preserved by $s_E$ and $t_E$.
- is an internal fibration in DblCat iff, in addition, $y_E$ and $\otimes_E$ preserve Cartesian morphisms.

Furthermore, a <u>strict</u> double functor $P$ is an internal fibration in $\underline{\mathbf{DblCat}}_s$ if and only if $P_0$ and $P_1$ are fibrations that admit cleavages that are preserved by all of $s_{\mathbb{E}}$, $t_{\mathbb{E}}$, $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$

# Double Indexing Functors

(Take 1)

# Double Indexing Functors

Note: (1.) Categories are monoids in Span (Set)

$$\begin{tikzcd}
C_0 \arrow[d, equal] & C_0 \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "y"] & C_0 \arrow[d, equal] \\
C_0 & C_1 \arrow[l, "s"] \arrow[r, "t"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & C_1 \times_{C_0} C_1 \arrow[dl] \arrow[dr] \arrow[dd, bend right, "\mu"] & & \\
C_0 \arrow[d, equal] & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 \arrow[d, equal] \\
C_0 & & C_1 \arrow[ll, "s"] \arrow[rr, "t"'] & & C_0
\end{tikzcd}$$

Unit $\longleftrightarrow$ identities for the cat$^y$

multiplication $\longleftrightarrow$ composition for the cat$^y$.

(2) Moeller and Vasilakopoulou used:

$\underline{Fib} \simeq \underline{ICat} \implies \underline{Ps\,Mon}(\underline{Fib}, \times) \simeq \underline{Ps\,Mon}(\underline{ICat}, \times)$

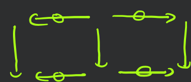
To generalize this further we need:

* double 2-categories ( pseudo categories in $\underline{\text{2-Cat}}$ )
* pseudo monoids in double 2-categories

<u>Result</u>: pseudo categories in a 2-cat$^y$ $\mathcal{C}$ correspond to pseudo monoids in $\mathcal{S}pan(\mathcal{C})$.

Recall that we want to take the source and target from a more restricted class of arrows, say $\Sigma$:

<u>Result</u>: pseudo cat$^s$ in $\mathcal{C}$ with $s, t$ in $\Sigma$ correspond to pseudo monoids in $\mathcal{S}pan_{\Sigma}(\mathcal{C})$.



# Double 2-Categories

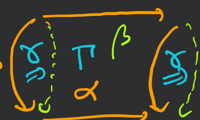
A double 2-category is a pseudo cat$^y$ in 2Cat

$$\begin{tikzcd} \mathcal{E}_1 \times_{\mathcal{E}_0} \mathcal{E}_1 \arrow[r, "\otimes"] & \mathcal{E}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathcal{E}_0 \arrow[l, "y"'] \end{tikzcd}$$

so we have

$$\begin{tikzcd} \bullet \arrow[r, "|" marking] & \bullet \end{tikzcd} \qquad \bullet \quad \text{objects}$$

$$\begin{tikzcd} \bullet \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[d] \\ \bullet \arrow[r, "|" marking] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd} \quad \text{arrows}$$

$\Gamma : \alpha \Rrightarrow \beta$ 


2-cells

# The Double 2-Category $\mathcal{S}pan(\underline{Cat})$

**<u>Objects</u>** categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$

**<u>Arrows</u>** functors

**<u>Proarrows</u>** spans $\mathcal{S} \xleftarrow{S} \mathcal{A} \xrightarrow{T} \mathcal{J}$

**<u>Dbl cells</u>** commutative diagrams

$$\begin{tikzcd}
\mathcal{S} \arrow[d, "G"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F"] & \mathcal{J} \arrow[d, "H"] \\
\mathcal{S}' & \mathcal{A}' \arrow[l, "S'"'] \arrow[r, "T'"] & \mathcal{J}'
\end{tikzcd}$$

# Dbl 3-cells

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "G"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F"] & \mathcal{T} \arrow[d, "H"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}
\qquad (\gamma, \varphi, \theta) \qquad \Longrightarrow
$$

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "G'"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F'"] & \mathcal{T} \arrow[d, "H'"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}
$$

Consists of

$$\begin{tikzcd}
\mathcal{S} \arrow[d, bend right=50, "\mathcal{G}"'] \arrow[d, dashed, bend left=50, "\mathcal{G}'"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, bend right=50, "F"'] \arrow[d, dashed, bend left=50, "F'"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathcal{T} \arrow[d, bend right=50, "H"'] \arrow[d, bend left=50, "H'"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}$$

such that the two cylinders commute.

# Double Indexing Functors

$\mathbb{I}\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$ is the slice category with

- objects: contravariant lax double pseudo functors $F : \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$
  ( $\mathbb{A}$ - indexed ps. dbl. categories )

- morphisms: $(H, \tau) : F \longrightarrow G$ :

$H$ lax dbl. pseudo functor

$\tau$: lax dbl. ps. ntl. trafo

$$\begin{tikzcd} \mathbb{A}^{op} \arrow[rr, "H^{op}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Rrightarrow \tau"] & \mathbb{B}^{op} \arrow[dl, "G"] \\ & \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}}) & \end{tikzcd}$$

# Lax Double Pseudo Functors

**Definition** A <u>lax double pseudo functor</u>

$$F : \mathbb{D} \longrightarrow \mathbb{E}$$

between dbl 2-categories $\mathbb{D}, \mathbb{E}$, consists of:

- Pseudo functors : $F_0 : \mathbb{D}_0 \longrightarrow \mathbb{E}_0$
  $F_1 : \mathbb{D}_1 \longrightarrow \mathbb{E}_1$

- Comparison pseudo natural transformations:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"'] & \mathbb{E}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\iota}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1
\end{tikzcd}$$

- Invertible associativity and unitor modifications

$$
\begin{tikzcd}
 & & \mathbb{D}_1^{(3)} \arrow[dll, "\mathbb{F}_1^{(3)}"'] \arrow[drr, "\otimes\times 1"] \arrow[dd] & & \\
\mathbb{E}_1^{(3)} \arrow[dd, "1\times\otimes"'] \arrow[dr, phantom, "1\times\phi"] & & & & \mathbb{D}_1^{(2)} \arrow[dd, "\otimes"] \arrow[dl, phantom, "\overset{a}{\cong}"] \\
 & & \mathbb{D}_1^{(2)} \arrow[dll] \arrow[drr] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & & \\
\mathbb{E}_1^{(2)} \arrow[drr, "\otimes"'] & & {} & & \mathbb{D}_1 \arrow[dll, "\mathbb{F}_1"] \\
 & & \mathbb{E}_1 & &
\end{tikzcd}
\quad \overset{\Phi}{\cong} \quad
\begin{tikzcd}
 & & \mathbb{D}_1^{(3)} \arrow[dll, "\mathbb{F}_1^{(3)}"'] \arrow[drr, "\otimes\times 1"] \arrow[d, phantom, "\overset{\phi\times 1}{\Rightarrow}"] & & \\
\mathbb{E}_1^{(3)} \arrow[drr] \arrow[dd, "1\times\otimes"'] \arrow[ddr, phantom, "\overset{a}{\cong}"] & & {} & & \mathbb{D}_1^{(2)} \arrow[dll] \arrow[dd, "\otimes"] \arrow[ddl, phantom, "\overset{\phi}{\Rightarrow}"] \\
 & & \mathbb{E}_1^{(2)} \arrow[dd] & & \\
\mathbb{E}_1^{(2)} \arrow[drr, "\otimes"'] & & & & \mathbb{D}_1 \arrow[dll, "\mathbb{F}_1"] \\
 & & \mathbb{E}_1 & &
\end{tikzcd}
$$

etc.

satisfying well-definedness and coherence conditions.

# Lax Double Pseudo Natural Transformations

A lax dbl ps. ntl transformation:

$$\tau : F \Rightarrow G : \mathbb{D} \rightrightarrows \mathbb{E}$$

Consists of:

- pseudo ntl transformations

$$\tau_0 : F_0 \Rightarrow G_0,\quad \tau_1 : F_1 \Rightarrow G_1$$

- modifications:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\tau_1 \times_{\tau_0} \tau_1}{\Longrightarrow}"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"] & \mathbb{E}_1
\end{tikzcd}
\overset{T}{\Rrightarrow}
\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\tau_1}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"'] & \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, equal] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\tau_0}{\Rightarrow}"] & \mathbb{D}_0 \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\overset{2}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_0 \arrow[r, equal] & \mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1
\end{tikzcd}
\quad \overset{\mathrm{I}}{\Rrightarrow} \quad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "y"] \arrow[d, "F_0"] \arrow[dr, phantom, "\overset{2}{\Rightarrow}"] & \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\tau_1}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1
\end{tikzcd}
$$

satisfying multiplicativity and unitality conditions.

We write $\underline{\mathrm{Dbl2Cat}}(\mathbb{D}, \mathbb{E})$ for the cat$^y$ of lax dbl pseudofunctors and lax dbl. ps. ntl. transformations.

# The Representation Theorem

# The Representation Theorem

<u>Theorem</u> (Cruttwell, Lambert, P., Szyld)

There is an equivalence of categories

$$\underline{\mathrm{DblFib}} \simeq \mathcal{I}\,\mathrm{Span}(\underline{\mathrm{Cat}})$$

<u>Idea for the proof:</u> use pseudo monoids in double 2-categories.

$$\underline{\mathrm{Fib}} \simeq \underline{\mathcal{I}\mathrm{Cat}} \rightsquigarrow \underline{c\mathrm{Fib}} \simeq \underline{\mathcal{I}\mathrm{Cat}}_t$$

so: $\mathrm{Span}_c(\underline{\mathrm{Fib}}) \simeq \mathrm{Span}_t(\underline{\mathcal{I}\mathrm{Cat}})$

$\mathrm{Span}_c(\underline{\mathrm{Fib}})$ has objects: $\begin{array}{c}\underline{E}\\ \downarrow p\\ \underline{B}\end{array}$ fibrations with cleavage

**Notation:**

$P$

$\mathrm{Span}_t(\underline{\mathrm{ICat}})$ has objects: $\underline{B}^{op} \xrightarrow{F} \underline{\mathrm{Cat}}$ ps. functor

$F$

arrows:

$$\begin{tikzcd} \underline{E} \arrow[r, "f^T"] \arrow[d, "p"'] & \underline{E}' \arrow[d, "p'"] \\ \underline{B} \arrow[r, "f^\perp"'] & \underline{B}' \end{tikzcd}$$

cartesian arrow preserving

$$P \xrightarrow{f} P'$$

arrows:

$$\begin{tikzcd} \underline{B}^{op} \arrow[rr, "H"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] & (\underline{B}')^{op} \arrow[dl, "F'"] \\ & \underline{\mathrm{Cat}} & \end{tikzcd}$$

$\theta$ pseudo trafo

$$F \xrightarrow{(H,\theta)} F'$$

Proarrows : $P \xleftarrow{\ell} Q \xrightarrow{r} R$ $\qquad \ell^{\top}, r^{\top}$ cleavage preserving

Proarrows : $F \xleftarrow{(L,\lambda)} \mathcal{G} \xrightarrow{(R,\rho)} K$ $\qquad \lambda, \rho$ strict natural transf$^{ns}$

Cells :

$$\begin{tikzcd}
P \arrow[d, "f"'] & Q \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "g"] & R \arrow[d, "h"] \\
P' & Q' \arrow[l, "\ell'"'] \arrow[r, "r'"] & R'
\end{tikzcd}$$

$\ell^{\top}, r^{\top}, \ell'^{\top}, r'^{\top}$ cleavage preserving

$f^{\top}, g^{\top}, h^{\top}$ Cartesian arrow pres.

Cells :

$$\begin{tikzcd}
F \arrow[d, "{(H,\theta)}"'] & \mathcal{G} \arrow[l, "{(L,\lambda)}"', "\bullet" marking] \arrow[r, "{(R,\rho)}", "\bullet" marking] \arrow[d] & K \arrow[d] \\
F' & \mathcal{G}' \arrow[l, "{(L',\lambda')}", "\bullet" marking] \arrow[r, "{(R',\rho')}"', "\bullet" marking] & K'
\end{tikzcd}$$

Then lift :

$$\underline{\mathrm{Dbl}\,\mathrm{Fib}} := \underline{\mathrm{Ps}\,\mathrm{Mon}}\left(\mathrm{Span}_c(\underline{\mathrm{Fib}})\right) \simeq$$

$$\simeq \underline{\mathrm{Ps}\,\mathrm{Mon}}\left(\mathrm{Span}_t(\mathbb{I}\underline{\mathrm{Cat}})\right) \simeq \mathbb{I}\mathrm{Span}(\underline{\mathrm{Cat}}).$$

# Connections with Known Constructions

$$
\begin{tikzcd}
\text{Monoidal Fibrations} \arrow[dr, "\text{monoids are cats}"'] & & \text{2-Fibrations} \arrow[dl, "\text{quintet constr.}"] \\
& \text{Double Fibrations} & \\
\text{Discrete Dbl Fib.} \arrow[ur] & & \text{Double Gr. Constr.} \arrow[ul]
\end{tikzcd}
$$

All of these have an indexed notion and a Grothendieck category of elements.

In our case the composition of the functors used in the proof gives us also a category of elements construction.

# The Double Category of Elements

# The Double Grothendieck Construction

Start with $F : \mathbb{D}^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathcal{C}at})$ :

$$F_0 : \mathbb{D}_0^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathcal{C}at})_0 = \underline{\mathcal{C}at} \qquad (1)$$

$$F_1 : \mathbb{D}_1^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathcal{C}at})_1$$

and a further induced functor :

$$\mathbb{D}_1^{op} \xrightarrow{F_1} \mathbb{S}\mathrm{pan}(\underline{\mathcal{C}at})_1 \xrightarrow{\mathrm{apx}} \underline{\mathcal{C}at} \qquad (2)$$

Apply the ordinary elements construction to (1) and (2):

$$\mathbb{E}l(F)_0 \longrightarrow \mathbb{D}_0 \qquad \mathbb{E}l(F)_1 \longrightarrow \mathbb{D}_1$$

cloven fibrations.

# Details of $\mathbb{E}\mathrm{l}(F)$

Some notation:

- for $m : A \not\to B$ in $\mathbb{D}$, write $Fm$ in $\mathrm{Span}(\underline{\mathrm{Cat}})$ as :

$$\begin{tikzcd} & Fm \arrow[dl, "L_m"'] \arrow[dr, "R_m"] & \\ FA & & FB \end{tikzcd}$$

- for $A \overset{m}{\not\to} B \overset{n}{\not\to} C$ in $\mathbb{D}$, we have the laxity comparison cell

$$\begin{tikzcd} FA \arrow[d, equal] & Fm \times_{FB} Fn \arrow[l] \arrow[r] \arrow[d, "\phi_{m,n}"] & FC \arrow[d, equal] \\ FA & F(m \otimes n) \arrow[l, "L_{m\otimes n}"'] \arrow[r, "R_{m\otimes n}"] & FC \end{tikzcd}$$

Now $\mathbb{E}\mathrm{l}(F)$ is given by:

- objects: $(C, x)$ $\quad C$ in $\mathbb{D}$, $x$ in $FC$.

- arrows: $(f, \bar{f}) : (C, x) \longrightarrow (D, y)$

  with $f : C \to D$ in $\mathbb{D}$

  and $\bar{f} : x \longrightarrow f^* y$ $\big(= F(f)(y)\big)$ in $FC$.

- proarrows: $(m, \bar{m}) : (C, x) \nrightarrow (D, y)$

  with $C \overset{m}{\nrightarrow} D$ in $\mathbb{D}$

  $\bar{m} \in Fm$ $\quad \left( FC \xleftarrow[L_m]{} Fm \xrightarrow[R_m]{} FD \right)$

  s.t. $L_m(\bar{m}) = x$

  $R_m(\bar{m}) = y$

• double cells:

$$\begin{tikzcd}
(A,x) \arrow[r, "{(m,\bar{m})}", "|" marking] \arrow[d, "{(f,\bar{f})}"'] \arrow[dr, phantom, "{(\theta,\bar{\theta})}"] & (B,y) \arrow[d, "{(g,\bar{g})}"] \\
(C,z) \arrow[r, "{(n,\bar{n})}"', "|" marking] & (D,w)
\end{tikzcd}$$

with

$$\begin{tikzcd}
A \arrow[r, "m", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\
C \arrow[r, "n"', "|" marking] & D
\end{tikzcd}$$

in $\mathbb{D}$

$$\begin{tikzcd}
FA \arrow[d, "Ff"'] & Fm \arrow[l, "L_m"'] \arrow[r, "R_m"] \arrow[d, "F\theta"] & FB \arrow[d, "Fg"] \\
FC & Fn \arrow[l, "L_n"] \arrow[r, "R_n"'] & FD
\end{tikzcd}$$

and $\bar{m} \xrightarrow{\bar{\theta}} \theta^* \bar{n}$ an arrow in $Fm$ s.t.

$$L_m(\bar{\theta}) = \bar{f} \quad \text{and} \quad R_m(\bar{\theta}) = \bar{g}.$$

Composition in the "arrow direction" is as expected:

- for arrows $(A,x) \xrightarrow{(f,\bar{f})} (B,y) \xrightarrow{(g,\bar{g})} (C,z)$

  the composite is $\left(gf, \phi_{f,g} f^*(\bar{g})\bar{f}\right) : (A,x) \longrightarrow (C,z)$

- for cells $(m,\bar{m}) \xRightarrow{(\theta,\bar{\theta})} (n,\bar{n}) \xRightarrow{(\delta,\bar{\delta})} (p,\bar{p})$

  the composite is

  $$\left(\delta\theta, \phi_{\theta,\delta}\theta^*(\bar{\delta})\bar{\theta}\right) : (m,\bar{m}) \Longrightarrow (p,\bar{p}).$$

- Units: $\left(1_C, (\varphi_C)_x\right) : (C,x) \longrightarrow (C,x)$

  $\left(1_m, (\varphi_m)_{\bar{m}}\right) : (m,\bar{m}) \longrightarrow (m,\bar{m}).$

$$\mathbb{E}l(F)_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{E}l(F)_0$$ are defined by

$$s(\theta, \bar{\theta}) = (f, \bar{f})$$

$$t(\theta, \bar{\theta}) = (g, \bar{g})$$

- proarrow composition:

for $(A, x) \xrightarrow{(m, \bar{m})} (B, y) \xrightarrow{(n, \bar{n})} (C, z)$

the composite is:

$$(m \otimes n, \phi_{m,n}(\bar{m}, \bar{n})) : (A, x) \longrightarrow (C, z)$$

- Composition for cells and proarrow units are given using appropriate components of the structure isos related to pseudo naturality.

# Results

- $\pi : \mathbb{E}l(F) \longrightarrow \mathbb{D}$ is a double fibration.
- This is the object part of an equivalence of categories

$$\underline{\mathrm{DblFib}} \simeq \mathbb{I}\,\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$$

which specializes to

$$\underline{\mathrm{DblFib}}(\mathbb{B}) \simeq \underline{\mathrm{Dbl2Cat}}(\mathbb{B}^{op}, \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}}))$$

for each dbl cat$^y$ $\mathbb{B}$.

# Examples

- Let $\underline{A}$ be a category with pushouts and $\mathbb{C}\mathrm{sp}(\underline{A})$ the double cat$^y$ with cells:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Z \arrow[d, "w"] & Y \arrow[l] \arrow[d, "v"] \\
X' \arrow[r] & Z' & Y' \arrow[l]
\end{tikzcd}$$

- For any lax double functor

$$F : \mathbb{C}\mathrm{sp}(\underline{A}) \to \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$$

the dbl cat$^y$ of elements $\mathbb{E}\mathrm{l}(F)$ is $F\text{-}\mathbb{C}\mathrm{sp}$, the double category of $F$-decorated cospans. [Patterson, 2023].

This slightly generalizes the decorated cospans from Baez-Courser-Vasilakopoulou.

# Double Indexing Functors

# (Take 2)

# How can we view $\mathbb{E}l(F)$ as a colimit?

- For this we need an indexing functor into a suitable double 2-category with double categories as objects.

- To guide us to the correct double 2-category observe:

pseudo dbl. functors $\underline{\mathbb{1}} \xrightarrow{F} \mathbb{S}pan(Cat)$ correspond to $\underline{C}_0 \xleftarrow{s} \underline{C}_1 \xrightarrow{t} \underline{C}_0$ with

$$\begin{tikzcd}
\underline{C}_0 \arrow[d, "1"'] & \underline{C}_0 \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, "y"] & \underline{C}_0 \arrow[d, "1"] \\
\underline{C}_0 & \underline{C}_1 \arrow[l, "s"'] \arrow[r, "t"] & \underline{C}_0
\end{tikzcd}$$

Note:

$\underline{C}_0 = F(*)$

$\underline{C}_1 = \mathrm{Apex}(F(id_*))$

and

$$
\begin{tikzcd}[column sep=small]
 & & & \underline{C_1} \times_{\underline{C_0}} \underline{C_1} \arrow[dl] \arrow[dr] \arrow[dd, "\otimes"] & & \\
\underline{C_0} \arrow[dr, equal] & \underline{C_1} \arrow[l, "s"'] \arrow[rr, "t"] & & \underline{C_0} & \underline{C_1} \arrow[l] \arrow[r] & \underline{C_0} \arrow[dl, equal] \\
 & \underline{C_0} & & \underline{C_1} \arrow[ll, "s"] \arrow[rr, "t"'] & & \underline{C_0}
\end{tikzcd}
$$

i.e. to double categories

$$
\begin{tikzcd}
\underline{C_1} \times_{\underline{C_0}} \underline{C_1} \arrow[r, "\otimes"] & \underline{C_1} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \underline{C_0} \arrow[l, bend right=40, "y"']
\end{tikzcd}
$$

$\underline{C_1} = \mathrm{Apex}(F(\mathrm{id}_x))$

$\underline{C_0} = F(x)$

- These double categories are the fibers of the double fibration

$$\mathbb{E}\mathrm{l}(F) \xrightarrow{\pi} \mathbb{A}.$$

- We can also use them to define a new indexing functor:

$$\tilde{F} : \mathbb{A} \longrightarrow \mathbb{P}\mathrm{rof}(\mathrm{Dbl\,Cat})$$

- $\mathbb{P}\mathrm{rof}(\mathrm{Dbl\,Cat})$ is the dbl 2-category with
  - $\mathbb{P}\mathrm{rof}(\mathrm{Dbl\,Cat})_0 = \underline{\mathrm{Dbl\,Cat}}$ with arrow-valued transformations as 2-cells
  - <u>pro arrows</u> are internal profunctors in <u>Cat</u>:

or: modules

$$\mathbb{D} \xrightarrow{m} \mathbb{E}$$

is given by a span of categories

$$\mathbb{D}_0 \xleftarrow{L_m} M \xrightarrow{R_m} \mathbb{E}_0$$

with associative actions

$$\mathbb{D}_1 \times_{\mathbb{D}_0} M \xrightarrow{\lambda} M \qquad M \times_{\mathbb{E}_0} \mathbb{E}_1 \xrightarrow{\rho} M$$

that commute with each other.

- double cells:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "m"] \arrow[d, "F"'] \arrow[dr, phantom, "\theta"] & \mathbb{E} \arrow[d, "G"] \\ \mathbb{D}' \arrow[r, "m'"'] & \mathbb{E}' \end{tikzcd}$$

are $\theta : M \to M'$ that commute with the actions.

- 3-cells are compatible triples of transformations.

# Transition of Indexing Functors

Given $F : \mathbb{A} \longrightarrow \mathrm{Span}(\underline{\mathrm{Cat}})$, we define

$$\tilde{F} : \mathbb{A} \longrightarrow \mathbb{P}\mathrm{rof}(\mathrm{Dbl\,Cat})$$

by:

- <u>objects</u>: $A \longmapsto \left( \begin{tikzcd} & F(id_A) \arrow[dl, "L_{id_A}"'] \arrow[dr, "R_{id_A}"] & \\ FA & & FA \end{tikzcd} \right) =: F(id_A) = \tilde{F}(A)$

  considered as a double cat$^y$.

- <u>arrows</u>: for an arrow $v : A \longrightarrow B$ in $\mathbb{A}$,

there is a dbl cell

$$\begin{tikzcd} A \arrow[r, "id_A", "\shortmid" marking] \arrow[d, "v"'] \arrow[dr, phantom, "id_v"] & A \arrow[d, "v"] \\ B \arrow[r, "id_B"', "\shortmid" marking] & B \end{tikzcd}$$

$F$ sends this to:

$$
\begin{tikzcd}
FA \arrow[d, "Fv"'] & F(id_A) \arrow[l, "L_{id_A}"'] \arrow[r, "R_{id_A}"] \arrow[d, "F(id_v)"] & FA \arrow[d, "Fv"] \\
FB & F(id_B) \arrow[l, "L_{id_B}"] \arrow[r, "R_{id_B}"'] & FB
\end{tikzcd}
$$

and this induces a ps. double functor

$$\tilde{F}(v) := F(id_v) : \tilde{F}(A) = F(id_A) \longrightarrow \tilde{F}(B) = F(id_B)$$

i.e., an arrow of $\mathrm{Prof}(\mathrm{Dbl\,Cat})$.

- pro-arrows: a proarrow $A \overset{m}{\nrightarrow} B$ in $\mathbb{A}$

  is sent by $F$ to:

$$
\begin{tikzcd}
FA & Fm \arrow[l, "L_m"'] \arrow[r, "R_m"] & FB
\end{tikzcd}
$$

With:

$$\begin{tikzcd}
 & & Fm \times_{FB} Fid_B \arrow[dl] \arrow[dr] \arrow[dd, bend right, "\rho_m"'] & & \\
FA & Fm \arrow[l, "Lm"'] \arrow[r, "Rm"] & FB & Fid_B \arrow[l, "Lid_B"'] \arrow[r, "Rid_B"] & FB \arrow[dl, equal] \\
FA \arrow[u, phantom] & FA \arrow[from=2-1, equal] & Fm \arrow[l, "Lm"] \arrow[r, "Rm"'] & FB &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & Fid_A \times_{FA} Fm \arrow[dl] \arrow[dr] \arrow[dd, bend right, "\lambda_m"'] & & \\
FA & Fid_A \arrow[l, "Lid_A"'] \arrow[r, "Rid_A"] & FA & Fm \arrow[l, "Lm"'] \arrow[r, "Rm"] & FB \arrow[dl, equal] \\
 & FA \arrow[from=2-1, equal] & Fm \arrow[l, "Lm"] \arrow[r, "Rm"'] & FB &
\end{tikzcd}$$

This gives $Fm$ the structure of a

Profunctor :

$$\tilde{F}(m) : \tilde{F}A \xrightarrow{\ \ \shortmid\ \ } \tilde{F}B .$$

- dbl cells: for

$$\begin{tikzcd} A \arrow[r, "m", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\ C \arrow[r, "n"', "\shortmid"{marking}] & D \end{tikzcd}$$

in $\mathbb{A}$

F gives:

$$\begin{tikzcd} FA \arrow[d, "Ff"'] & Fm \arrow[l, "L_m"'] \arrow[r, "R_m"] \arrow[d, "F\theta"] & FB \arrow[d, "Fg"] \\ FC & Fn \arrow[l, "L_n"] \arrow[r, "F_n"'] & FD \end{tikzcd}$$

and $F\theta : Fm \longrightarrow Fn$ is equivariant with respect to the actions defined before, so $\tilde{F}\theta = F\theta$ fits in :

$$
\begin{tikzcd}
\tilde{F}A \arrow[r, "\tilde{F}m", "/"{marking}] \arrow[d, "\tilde{F}f"'] \arrow[dr, phantom, "\tilde{F}\theta \Downarrow"] & \tilde{F}B \arrow[d, "\tilde{F}g"] \\
\tilde{F}C \arrow[r, "\tilde{F}n", "/"{marking}] & \tilde{F}D
\end{tikzcd}
$$

lots of details to check ... and we have:

$$
\Uparrow \Downarrow \quad \frac{F : \mathbb{A} \longrightarrow \mathrm{Span}(\underline{\mathrm{Cat}})}{\tilde{F} : \mathbb{A} \longrightarrow \mathbb{P}\mathrm{rof}(\mathrm{DblCat})}
$$

"$\Longrightarrow$" is given by forgetting the arrow parts.

<u>to do</u>: make this an equivalence of categories.

# $\mathbb{E}\mathrm{l}(F)$ as a cone for $\tilde{F}$

- for $A$ in $\mathbb{A}$:

$$\tilde{F}(A) = \begin{tikzcd} F(id_A) \arrow[d, shift left] \arrow[d, shift right] \arrow[rr, "\varphi_A", "\text{ps. dbl. functor}"'] & & \mathbb{E}\mathrm{l}(F)_1 \arrow[d, shift left] \arrow[d, shift right] \\ FA & & \mathbb{E}\mathrm{l}(F)_0 \end{tikzcd}$$

* <u>obj</u>: $x \in \mathrm{Obj}(FA) \longmapsto (A, x)$

* <u>arrows</u>: $x \xrightarrow{v} x'$ in $\mathrm{Arr}(FA)$ $\longmapsto (1_A, (\varphi_A)_{x'} \circ v)$

* <u>proarrows</u>: $m \in \mathrm{Obj}(F(id_A)) \longmapsto (id_A, m)$

* <u>cells</u>: $\theta \in \mathrm{Arr}(F(id_A)) \longmapsto (2_A, (\varphi_{id_A})_t \circ \theta)$

- for $A \overset{m}{\nrightarrow} B$ in $\mathbb{A}$, we need a dbl cell

$$\begin{tikzcd}
\tilde{F}A \arrow[r, "\tilde{F}m", "/"{marking}] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\eta_m"] & \tilde{F}B \arrow[d, "\eta_B"] \\
\mathbb{E}\ell(F) \arrow[r, "/"{marking}, "\mathrm{id}_{\mathbb{E}\ell(F)}"'] & \mathbb{E}\ell(F)
\end{tikzcd}$$

i.e.

$$\begin{tikzcd}
FA \arrow[d, "(\eta_A)_0"'] & Fm \arrow[l] \arrow[r] \arrow[d, "\eta_m"] & FB \arrow[d, "(\eta_B)_0"] \\
\mathbb{E}\ell(F)_0 & \mathbb{E}\ell(F)_1 \arrow[l, "s"'] \arrow[r, "t"] & \mathbb{E}\ell(F)_0
\end{tikzcd}$$

satisfying additional conditions.

- on objects of $Fm$ : $\overline{m} \longmapsto (m, \overline{m})$
- on arrows of $Fm$ : $(\overline{m} \xrightarrow{\theta} \overline{m}') \mapsto (1_m, (\eta_m)_{\overline{m}'} \circ \theta)$

We can then check that this is indeed a cone:

$$
\begin{tikzcd}
\cdot \arrow[r, "\tilde{F} m", "/" marking] \arrow[d] \arrow[dr, phantom, "\tilde{F}\theta"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\tilde{F} n"', "/" marking] \arrow[d] \arrow[dr, phantom, "\pi_n"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\mathrm{id}_{El(F)}"', "/" marking] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r, "\tilde{F} m", "/" marking] \arrow[d] \arrow[dr, phantom, "\pi_m"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\mathrm{id}_{El(F)}"', "/" marking] & \cdot
\end{tikzcd}
$$

and a suitable equation for

$$\pi_{m_1 \otimes m_2} \; \text{- - -}$$

## The Universal Property of this Cone

To be done — this is an invitation!

(there is the usual double categorical colimit property as described in Grandis-Paré, but there are also higher dimensional aspects)

Potential applications:

— understand how various known constructions such as decorated cospans are colimits

— this can help with determining various ways of describing morphisms between them.