Profunctors and Promonoidal Categories

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Motivation

One of the central aims of category theory is to abstract structure that is usually considered as structure on sets to more general contexts.

Some examples include:

- monoidal categories and enriched categories as generalisations of abelian categories
- toposes as generalisations of Set
- Grothendieck fibrations as generalisations of the family fibration $Fam(C)$ over Set

Motivation

Profunctors generalise functors.

Profunctors have many names:

- Relators
  - relations generalise functions, relators generalise functors
- Distributors
  - functional analysis, distributions generalise functions
- (Bi)Modules
  - Bim is the subcategory of $Prof(Ab)$ whose 0-cells are one-object Ab-categories

Promonoidal categories are like monoidal categories.
However, we **replace functors by profunctors**.

Profunctors

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. A **profunctor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor

$$P : \mathcal{D}^{op} \times \mathcal{C} \to Set$$

denoted $P : \mathcal{C} \nrightarrow \mathcal{D}$. For $D \in \mathcal{D}$ and $C \in \mathcal{C}$, the set $P(D, C)$ is called the set of heteromorphisms from D to C.

We can also replace Set by any complete, co-complete, closed, symmetric, monoidal category $\mathcal{V}$ (Bénabou cosmos).

In this case, the profunctors are $\mathcal{V}$-functors $\mathcal{D}^{op} \otimes \mathcal{C} \to \mathcal{V}$.

Profunctors

Example

The identity profunctor $Id_{\mathcal{C}} : \mathcal{C} \nrightarrow \mathcal{C}$ is given by the hom functor $hom_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to Set$.

Theorem

Every functor $F : \mathcal{C} \to \mathcal{D}$ induces a profunctor $hom_{\mathcal{D}}((-), F(-)) : \mathcal{C} \nrightarrow \mathcal{D}$. We call these the representable profunctors.

In this case the set of heteromorphisms is $hom_{\mathcal{D}}(D, F(C))$.

Similarly, F induces a corepresentable profunctor $hom_{\mathcal{D}}(F(-), (-)) : \mathcal{D} \nrightarrow \mathcal{C}$.

The Bicategory $Prof$

Given two $\mathcal{V}$-profunctors $F : \mathcal{A} \nrightarrow \mathcal{B}$ and $G : \mathcal{B} \nrightarrow \mathcal{C}$, the composition $G \circ F : \mathcal{A} \nrightarrow \mathcal{C}$ is given by the co-end

$$(G \circ F)(C, A) := \int^{B \in \mathcal{B}} F(B, A) \otimes G(C, B)$$

Definition

The bicategory $Prof(\mathcal{V})$ has

- 0-cells: $\mathcal{V}$-categories $\mathcal{C}, \mathcal{D}, \dots,$
- 1-cells: $\mathcal{V}$-profunctors $P : \mathcal{C} \nrightarrow \mathcal{D}$,
- 2-cells: natural transformations between profunctors

Profunctors as Generalized Relations

Let $\mathcal{C}$ and $\mathcal{D}$ be enriched over the discrete category $\mathbf{2} = (\{0,1\}, <)$.
Then $\mathcal{C}$ and $\mathcal{D}$ are preorders.

A profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is a **2**-functor $P : \mathcal{D}^{op} \times \mathcal{C} \to \{0,1\}$.
P is a monotone map that corresponds to a relation $P \subseteq \mathcal{D} \times \mathcal{C}$.

Composition of (ordinary) relations:

For $S \subseteq X \times Y$ and $R \subseteq Y \times Z$, we have

$$(R \circ S)(x, z) \iff \exists\, y \in Y.\ S(x, y)\ \wedge\ R(y, z)$$

Profunctors as Generalized Relations

Composition in $Prof(\mathcal{V})$:

For $F : \mathcal{X} \nrightarrow \mathcal{Y}$ and $G : \mathcal{Y} \nrightarrow \mathcal{Z}$, we have

$$(G \circ F)(Z, X) \quad = \quad \int^{Y \in \mathcal{Y}} F(Y, X) \otimes G(Z, Y)$$

Viewing profunctors as generalised relations is **generalised logic**.
The co-end is a generalised existential.
The $\otimes$ is a generalised conjunction.
The heteromorphisms $F(X, Y)$ are types whose terms are proofs that X and Y are in a "generalised relation".

Motivation for Promonoidal Categories

A monoidal category is a category with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, satisfying associativity and unitarity conditions.

Since profunctors generalise functors, it makes sense to consider a category equipped with a binary profunctor satisfying analogous conditions.

From this point on, we are working only in $\mathcal{V} = Set$, but these results hold more generally.

Promonoidal Categories

Definition

A **promonoidal category** is a small category $\mathcal{C}$ with

- profunctor $P : \mathcal{C} \times \mathcal{C} \nrightarrow \mathcal{C}$, that is, $P : \mathcal{C}^{op} \times (\mathcal{C} \times \mathcal{C}) \to Set$
- profunctor $J : 1 \nrightarrow \mathcal{C}$, that is, $J : \mathcal{C}^{op} \to Set$
- $\alpha_{ABCD} : \int^X P(X,(B,C)) \otimes P(D,(A,X)) \cong \int^X P(X,(A,B)) \otimes P(D,(X,C))$
- $\lambda_{AB} : \int^X J(X) \otimes P(B,(A,X)) \cong hom_{\mathcal{C}}(B,A)$
- $\rho_{AB} : \int^X J(X) \otimes P(B,(X,A)) \cong hom_{\mathcal{C}}(B,A)$

satisfying the triangle and pentagon identities for promonoidal categories.

"Triangle" Identity

$$\begin{tikzcd}
J(Y) \circ (P_{XBY} \circ P_{AXC}) \arrow[rr, "id_{J(Y)} \cdot \alpha"] & & J(Y) \circ (P_{XYC} \circ P_{ABX}) \arrow[d, "\cong"] \\
(J(Y) \circ P_{XBY}) \circ P_{AXC} \arrow[u, "\cong"] & & (J(Y) \circ P_{XYC}) \circ P_{ABX} \arrow[d, "\lambda_{XC} \cdot id_{P_{ABX}}"] \\
hom(X, B) \circ P_{AXC} \arrow[u, "\rho^{-1}_{XB} \cdot id_{P_{ABX}}"] & & hom(X, C) \circ P_{ABX} \arrow[dl] \\
& P_{ABC} \arrow[ul] &
\end{tikzcd}$$

"Pentagon" Identity

$$
\begin{tikzcd}
P_{XCD} \circ (P_{YBX} \circ P_{AYE}) \arrow[rr, "id_{P_{XCD}} \cdot \alpha"'] & & P_{XCD} \circ (P_{YXE} \circ P_{ABY}) \arrow[rr, "\cong"'] & & (P_{XCD} \circ P_{YXE}) \circ P_{ABY} \arrow[d, "\alpha \cdot id_{P_{ABY}}"'] \\
(P_{XCD} \circ P_{YBX}) \circ P_{AYE} \arrow[u, "\cong"'] & & & & (P_{XDE} \circ P_{YCX}) \circ P_{ABY} \arrow[d, "\cong"'] \\
(P_{XBC} \circ P_{YXD}) \circ P_{AYE} \arrow[u, "\alpha \cdot id_{P_{AYE}}"'] \arrow[d, "\cong"'] & & & & P_{XDE} \circ (P_{YCX} \circ P_{ABY}) \arrow[d, equal] \\
P_{XBC} \circ (P_{YXD} \circ P_{AYE}) \arrow[d, "id_{P_{XBC}} \cdot \alpha"'] & & & & P_{YDE} \circ (P_{XCY} \circ P_{ABX}) \\
P_{XBC} \circ (P_{YDE} \circ P_{AXY}) \arrow[dr, "\cong"'] & & & & P_{YDE} \circ (P_{XBC} \circ P_{AXY}) \arrow[u, "id_{P_{YDE}} \cdot \alpha"'] \\
& (P_{XBC} \circ P_{YDE}) \circ P_{AXY} \arrow[rr, "\cong"] & & (P_{YDE} \circ P_{XBC}) \circ P_{AXY} \arrow[ur, "\cong"] &
\end{tikzcd}
$$

Monoidal Categories

Theorem

Every monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is also promonoidal.

We define

$$P(A, B, C) := hom_{\mathcal{C}}(A, B \otimes C)$$

$$J(A) := hom_{\mathcal{C}}(A, I)$$

That is, P and J are representable profunctors of $\otimes$ and I respectively.

Day Convolution

The functor category on a monoidal category inherits a monoidal structure.

Definition

For $(\mathcal{C}, \otimes, 1)$ a $\mathcal{V}$-enriched monoidal category, the **Day convolution** tensor product on the functor category $[\mathcal{C}, \mathcal{V}]$

$$\otimes_{Day} : [\mathcal{C}, \mathcal{V}] \otimes_{\mathcal{V}} [\mathcal{C}, \mathcal{V}] \to [\mathcal{C}, \mathcal{V}]$$

is given by the co-end

$$F \otimes_{Day} G : C \mapsto \int^{X,Y \in \mathcal{C}} hom_{\mathcal{C}}(X \otimes_{\mathcal{C}} Y, C) \otimes_{\mathcal{V}} F(X) \otimes_{\mathcal{V}} G(Y)$$

Day Convolution

A promonoidal $\mathcal{C}$ is sufficient to induce a monoidal structure on $[\mathcal{C}^{op}, \mathcal{V}]$.

Definition

For $(\mathcal{C}, P, J, \alpha, \lambda, \rho)$ a $\mathcal{V}$-enriched promonoidal category, the **Day convolution** tensor product on $[\mathcal{C}^{op}, \mathcal{V}]$

$$\otimes_{Day} : [\mathcal{C}^{op}, \mathcal{V}] \otimes_{\mathcal{V}} [\mathcal{C}^{op}, \mathcal{V}] \to [\mathcal{C}^{op}, \mathcal{V}]$$

is given by the co-end

$$F \otimes_{Day} G : C \mapsto \int^{X,Y \in \mathcal{C}^{op}} P(C, X, Y) \otimes_{\mathcal{V}} F(X) \otimes_{\mathcal{V}} G(Y)$$

There is an equivalence between the category of promonoidal structures on $\mathcal{C}$ with strong-promonoidal functors and the category $[\mathcal{C}^{op}, \mathcal{V}]$.

Multicategories

Definition

A **multicategory** $\mathcal{C}$ consists of

- A collection of objects, $\mathcal{C}_0$
- A collection of multimorphisms, $\mathcal{C}_1$
- A source map $s : \mathcal{C}_1 \to (\mathcal{C}_0)^*$ and a target map $t : \mathcal{C}_1 \to \mathcal{C}_0$

Identity Law: map $id_{(-)} : \mathcal{C}_0 \to \mathcal{C}_1$, given by $c \mapsto (id_c : c \to c)$

Composition Law: for every $f : c_1, \ldots, c_n \to c$ with an n-tuple $(f_1 : \bar{c_1} \to c_1, \ldots, f_n : \bar{c_n} \to c_n)$, we have a composite

$$f \circ (f_1, \ldots, f_n) : \bar{c_1}, \ldots, \bar{c_n} \to c$$

Multicategories

Theorem

A promonoidal structure on $\mathcal{C}$ can be identified with a multicategory structure on $\mathcal{C}^{op}$.

Let $\mathcal{C}$ be a promonoidal category.
We can define a co-multicategory $\tilde{\mathcal{C}}$ with:

Objects:
same as $\mathcal{C}$

Co-multimorphisms:
for $n \geq 0$, we have

$$Hom_{\mathcal{C}}(b;\ ) = J(b)$$

$$Hom_{\mathcal{C}}(b; a_1, \ldots, a_{n+1}) = \int^{x \in \mathcal{C}} Hom_{\mathcal{C}}(x; a_1, \ldots, a_n) \otimes P(b, x, a_{n+1})$$

Next Steps

- Linearly Distributive Promonoidal Categories
- Pre and Promonoidal Structure of Spacetime (Hefford, Kissinger)

Resources

- Distributors at Work, *Jean Bénabou*
- Coend Calculus, *Fosco Loregian*
- On Closed Categories of Functors, *Brian Day*
- Day Convolution for Monoidal Bicategories, *Alexander Corner*
- Metric Spaces, Generalized Logic, and Closed Categories, *F. William Lawvere*
- Kan Extensions Along Promonoidal Functors, *Brian Day, Ross Street*

- On the Pre- and Promonoidal Structure of Spacetime, *James Hefford, Aleks Kissinger*