A CARTESIAN EQUIPMENT BESTIARY

Michael Lambert

June 2023

FMCS 2023

GOALS OF THE TALK

- Unpack the notion of a *cartesian double category*
- Discuss Examples
- Study a bit the notion of *discreteness*
- Convince you that cartesian double categories are interesting and useful
- Exhibit a double category of pointless spaces

SOME HISTORY

CARTESIAN BICATEGORIES

- Carboni & Walters, *Cartesian Bicategories* (1987) [CW87]
  1. When is a monoidal bicategory recognizably *cartesian*?
  2. Examples: orders, partial maps, relations.
  3. Which bicategories are $\simeq$ **Rel**($\mathscr{E}$) for regular $\mathscr{E}$?
  4. Centrality of functional completeness and *discreteness*.
- Carboni et. al., *Cartesian Bicategories II* (2008) [CKWW08]
  1. Extend previous considerations to bicategories not necessarily locally posetal (e.g. profunctors, spans)
  2. Concept *groupoidal* replaces discreteness.

CARTESIAN DOUBLE CATEGORIES

- Eva Aleiferi, *Cartesian Double Categories* (2018) [Ale18]
    1. Define the notion in the spirit of [CKWW08].
    2. Which double cats are $\simeq \mathbb{S}\mathbf{pan}(\mathscr{E})$ for fin. compl. $\mathscr{E}$?
- M.L., *Double Categories of Relations* (2022) [Lam22]
    1. Which double categories are $\simeq \mathbb{R}\mathbf{el}(\mathscr{E})$ for regular $\mathscr{E}$?
    2. Equipment structure needed: The horizontal bicategory of any cartesian equipment is a cartesian bicategory.
    3. Consequently, discreteness as in [CW87].
    4. Functional completeness in terms of tabulators.

FURTHER COMMENTS ON LITERATURE

- Carboni & Street, *Order Ideals in Categories* [CS86]
    1. 2-category of orders and order-preserving maps
    2. Bicategory of orders and *ideals*
    3. These combine to form a double category. Is it cartesian?
- Number of interesting subsidiary results in [CW87]
    1. Discrete objects in monoids in semilattices are Joyal and Tierney's *discrete spaces* [JT84, §4.2]
    2. Discrete objects in the bicategory of orders and ideals are precisely equivalence relations [CW87, Example 2.3(ii)].

DOUBLE CATEGORY BASICS

DEFINITION

A **double category** $\mathbb{D}$ is a (pseudo) category in **Cat**.

1. $\mathbb{D}_0$ is the category of objects and ordinary arrows $f\colon A \to B$.
2. $\mathbb{D}_1$ is the category of proarrows $M\colon A \nrightarrow B$ and cells:

$$\begin{array}{ccc} A & \xrightarrow{\;M\;} & B \\ {\scriptstyle f}\downarrow & \phi & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\;N\;]{} & D \end{array}$$

3. source, target, unit functors src, tgt$\colon \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$, $y\colon \mathbb{D}_0 \to \mathbb{D}_1$
4. external composition for proarrows $M \otimes N$ and cells $\phi \otimes \psi$ (diagrammatic, associative up to coherent iso)

EXAMPLES

Old friends:

- $\mathbb{Q}(\mathcal{K})$ - quintets in $\mathcal{K}$ a 2-category
- $(\mathscr{V}, \otimes)$ - one object double category on a monoidal category
- $\mathcal{B}$ - bicategory as a double category with only identity arrows
- $\mathbb{S}$**pan** - sets, functions, spans
- $\mathbb{R}$**el** - sets, functions, relations
- $\mathbb{M}$**at**$(\mathscr{V})$ - sets, functions, and $\mathscr{V}$-matrices
- $\mathbb{R}$**ing** - rings (with unit), (unital) homomorphisms, bimodules
- $\mathbb{P}$**rof** - categories, functors, profunctors
- $\mathbb{M}$**et** - (Lawvere) metric spaces and metric profunctors

SPANS IN SOME DETAIL

- objects are sets
- morphisms are set functions $f : X \to Y$
- a proarrow $X \nrightarrow Y$ is a span $X \leftarrow S \to Y$
- a cell is a *span morphism*

$$\begin{array}{ccccc} X & \longleftarrow & S & \longrightarrow & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle g} \\ W & \longleftarrow & T & \longrightarrow & W \end{array}$$

- external units are identity spans $X \xleftarrow{1_X} X \xrightarrow{1_X} X$

SPANS IN FURTHER DETAIL

Span composition is by pullback

$$\begin{array}{ccccc} S \times_Y T & \dashrightarrow & T & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow & & \\ S & \longrightarrow & Y & & \\ \downarrow & & & & \\ X & & & & \end{array}$$

Associators induced by universal property of pullbacks; these satisfy a coherence condition.

EXAMPLES CONTINUED

New (?) acquaintances:

- $\mathbb{2}$ - ordinal double category
- $\mathbb{I}\mathbf{dl}(\mathscr{E})$ - orders and ideals in regular $\mathscr{E}$
- $\mathbb{S}\mathbf{lat}$ - semilatices and modules
- $\mathbb{F}\mathbf{rame}$ - frames and modules
- $\mathbb{L}\mathbf{oc}$ - locales and modules
- $\mathbb{D}\mathbf{Esp}$ - 2-/double spaces and modules
- $\mathbb{M}\mathbf{od}(\mathbb{D})$ - monoids and modules for suitable $\mathbb{D}$

More on these in due course.

DEFINITION [SHU08, §4]

A double category $\mathbb{D}$ is an **equipment** if the source-target projection $\langle \mathrm{src}, \mathrm{tgt} \rangle \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is equivalently

- a fibration
- an opfibration
- a bifibration.

Elsewhere these have been called *framed* bicategories [Shu08], *fibrant* double categories [Ale18] and *gregarious* double categories [DPP10]. The name *equipment* recalls Richard Wood's *proarrow equipment* [Woo82]

NICHES AND CONICHES

1. every *niche* completes to a cartesian cell

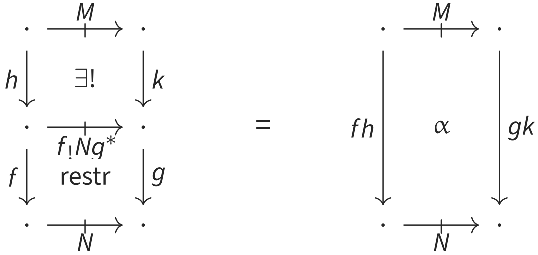
$$
\begin{array}{ccc}
A & & B \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
C & \xrightarrow[N]{} & D
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
A & \xrightarrow{f_! N g^*} & B \\
{\scriptstyle f}\big\downarrow & \text{restr} & \big\downarrow{\scriptstyle g} \\
C & \xrightarrow[N]{} & D
\end{array}
$$

2. every *coniche* completes to an opcartesian cell

$$
\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
C & & D
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f}\big\downarrow & \text{ext} & \big\downarrow{\scriptstyle g} \\
C & \xrightarrow[f^* M g_!]{} & D
\end{array}
$$

CARTESIAN UNIVERSAL PROPERTY

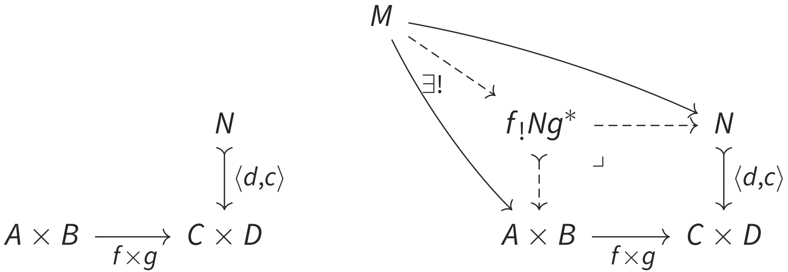
For any such cell α, there exists a unique globular cell satisfying:



Dually for extension cells.

EXAMPLE

In relations, a niche is equivalently a corner as at left:



The restriction is formed as the pullback, and *cartesian* is just its universal property. Thus, restrictions are *limit-like*.

COMPANIONS AND CONJOINTS [GP04, §1.2, 1.3]

Special cases are so-called *companions* $f_!$ and *conjoints* f^*

$$
\begin{array}{ccc}
A & \xrightarrow[]{f_!} & B \\
{\scriptstyle f}\downarrow & \text{restr} & \| \\
B & \xrightarrow[y_B]{} & B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow[]{y_A} & A \\
\| & \text{ext} & \downarrow{\scriptstyle f} \\
A & \xrightarrow[f_!]{} & B
\end{array}
$$

$$
\begin{array}{ccc}
B & \xrightarrow[]{f^*} & A \\
\| & \text{restr} & \downarrow{\scriptstyle f} \\
B & \xrightarrow[y_B]{} & B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow[]{y_A} & A \\
{\scriptstyle f}\downarrow & \text{ext} & \| \\
B & \xrightarrow[f^*]{} & A
\end{array}
$$

satisfying equations. Note: all restrictions and extensions constructed from companions and conjoints [Shu08, Theorem 4.1]

COMPANIONS AND CONJOINTS EXAMPLES

- Companion cells in $\mathbb{S}$**pan**:

$$\begin{array}{ccccc} X & = & X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ Y & = & Y & = & Y \end{array} \qquad\qquad \begin{array}{ccccc} X & = & X & = & X \\ \| & & \| & & \downarrow{\scriptstyle f} \\ X & = & X & \xrightarrow[f]{} & Y \end{array}$$

- Companion and conjoint proarrows in $\mathbb{R}$**el**

$$X \xrightarrow{\langle 1, f\rangle} X \times Y \qquad\qquad X \xrightarrow{\langle f, 1\rangle} Y \times X$$

the graph and opgraph of given set function $f\colon X \to Y$

EQUIPMENT EXAMPLES

See [GP04] and [Shu08] for 4/5 of these:

- $\mathbb{S}$**pan**
- $\mathbb{R}$**el**
- $\mathbb{P}$**rof**
- $\mathbb{R}$**ing** - [Par21, §2] for companions and conjoints
- $\mathbb{M}$**at**$(\mathscr{V})$
- others we’ll see in due course

NONEXAMPLES [GP04]

1. In quintets, not every arrow has a conjoint.
2. Let $\mathbb{D}\mathbf{bl}$ denote the double category whose
   - objects are double categories
   - arrows are lax functors
   - proarrows are oplax functors
   - cells are as defined in the reference

   Then a lax double functor has a companion in $\mathbb{D}\mathbf{bl}$ if, and only if, it is pseudo [GP04, Theorem 4.2]

CARTESIAN DOUBLE CATEGORIES

DEFINITION

A category $\mathscr{E}$ is **cartesian** if the functors $\mathscr{E} \to 1$ and $\Delta\colon \mathscr{E} \to \mathscr{E} \times \mathscr{E}$ each have right adjoints.

The unit of $\Delta \dashv \times$ gives the internal diagonals:

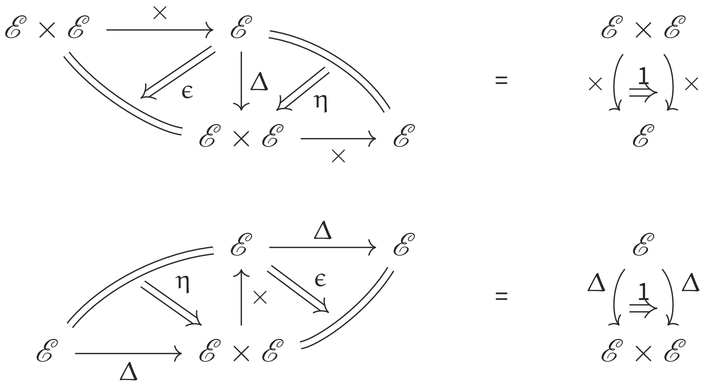
$$\eta_X\colon X \to X \times X$$

while the components of the counit maps are the projections:

$$\epsilon_{X,Y} = (\pi_X, \pi_Y)\colon (X \times Y, X \times Y) \to (X \times Y)$$

DIAGRAMMATIC FORMULATION

$\mathscr{E}$ has binary products if, and only if, both equations hold:



These make sense in any 2-category. (Likewise for terminals.)

TWO 2-CATEGORIES

- $\mathbf{Dbl}_l$ denotes the 2-category of double categories, lax functors, and transformations
- **Dbl** denotes the 2-category of double categories, pseudo double functors and transformations

DEFINITIONS [ALE18, §4.1, §4.2]

A double category $\mathbb{D}$ is **precartesian** if the double functors $!\colon \mathbb{D} \to 1$ and $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ have right adjoints in $\mathbf{Dbl}_l$.

A double category $\mathbb{D}$ is **cartesian** if it is precartesian and the right adjoints are pseudo (that is, if Δ and ! have right adjoints in **Dbl**.)

What does it mean to be cartesian, practically?

DOUBLE ADJUNCTIONS [GP04, §3.1]

Adjunction between double functors $F \dashv G$ (with G potentially lax) means (minimally)

- $F_0 \dashv G_0$
- $F_1 \dashv G_1$

in a double-categorically coherent way.

Consequently, $\mathbb{D}$ (pre)cartesian implies $\mathbb{D}_0$ and $\mathbb{D}_1$ each have finite products (in a coherent way i.e. laxity and transformation properties of units and counits).

CONSTRUCTION [ALE18, PROP 4.3.2]

Every cartesian equipment has *finite products locally* i.e. each hom category $\mathbb{D}(A, B)$ has finite products.

Constructed as restrictions:

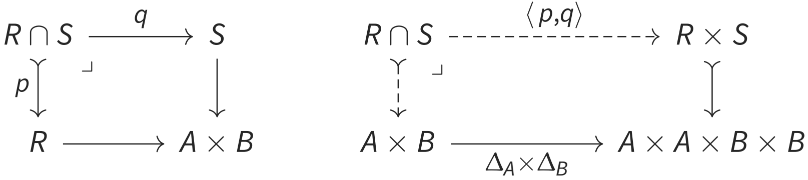
$$\begin{array}{ccc} A & \xrightarrow{\top} & B \\ {\scriptstyle !}\big\downarrow & \text{restr} & \big\downarrow{\scriptstyle !} \\ 1 & \xrightarrow[y_1]{} & 1 \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{M \wedge N} & B \\ {\scriptstyle \Delta}\big\downarrow & \text{restr} & \big\downarrow{\scriptstyle \Delta} \\ A \times A & \xrightarrow[M \times N]{} & B \times B \end{array}$$

Projections are given by composing with given projection cells in $\mathbb{D}_1$.

This generalizes the formula for local products in [CW87].

EXAMPLE: LOCAL PRODUCTS IN RELATIONS

Form the intersection of the monics as on the left:



The square on the right then presents the restriction.

PROPOSITION [ALE18, PROPS. 3.4.13, 3.4.16, 4.1.2]

Suppose that

- $\mathbb{D}$ is an equipment
- $\mathbb{D}_0$ has finite products
- $\mathbb{D}$ has finite products locally (each cat $\mathbb{D}(A, B)$ has fin prods).

The category $\mathbb{D}_1$ then has finite products and the assignments $1 \to \mathbb{D}$ and $\mathbb{D} \times \mathbb{D} \to \mathbb{D}$ defined via these products are in fact lax double functors right adjoint to ! and Δ respectively, i.e. $\mathbb{D}$ is precartesian.

PROPOSITION [ALE18, COR. 4.3.3]

Suppose that

- $\mathbb{D}$ is an equipment
- $\mathbb{D}_0$ has finite products
- $\mathbb{D}$ has finite products locally
- the resulting lax functors $1 \to \mathbb{D}$ and $\mathbb{D} \times \mathbb{D} \to \mathbb{D}$ are pseudo.

The double category $\mathbb{D}$ is then a cartesian equipment.

DEFINITION [GP99, APPENDIX]

A **lax functor** $F\colon \mathbb{X} \to \mathbb{D}$ consists of two functors $F_0\colon \mathbb{X}_0 \to \mathbb{D}_0$ and $F_1\colon \mathbb{X}_1 \to \mathbb{D}_1$ together with *laxity comparison cells*

$$
\begin{array}{ccc}
FX & \xrightarrow{\;y_{FX}\;} & FX \\
\| & F_X & \| \\
FX & \xrightarrow[F(y_X)]{} & FX
\end{array}
\qquad\qquad
\begin{array}{ccccc}
FX & \xrightarrow{\;FM\;} & FY & \xrightarrow{\;FN\;} & FZ \\
\| & & F_{M,N} & & \| \\
FX & & \xrightarrow[F(M\otimes N)]{} & & FZ
\end{array}
$$

satisfying a number of equations.

oplax means the comparisons point the other way

pseudo means that each F_X and $F_{M,N}$ is invertible

EXAMPLES

- lax functors $1 \to \mathbb{S}\mathbf{pan}$ are precisely small categories
- representables $y\colon \mathbb{D}^{op} \to \mathbb{S}\mathbf{pan}$ are in general lax [Par11]
- $\mathrm{Ob}\colon \mathbb{P}\mathbf{rof} \to \mathbb{S}\mathbf{pan}$ taking the set of objects of a small category is lax [Par11, §1.2]
- $\mathbf{Mon}\colon \mathbb{M}\mathbf{Cat} \to \mathbb{P}\mathbf{rof}$ taking a monoidal category to the category of monoids in it [GP04, §2.3, 2.4]

(PRE)CARTESIAN COMPARISON CELLS

If $\mathbb{D}$ is (pre)cartesian, the (lax) functor $\times\colon \mathbb{D}\times\mathbb{D}\to\mathbb{D}$ comes with comparison cells

$$
\begin{array}{ccc}
A\times B & \xrightarrow{y_{A\times B}} & A\times B \\
\| & \phi_{A,B} & \| \\
A\times B & \xrightarrow[y_A\times y_B]{} & A\times B
\end{array}
\qquad
\begin{array}{ccccc}
A\times B & \xrightarrow{M\times N} & C\times D & \xrightarrow{P\times Q} & E\times F \\
\| & & \phi_{(M,N),(P,Q)} & & \| \\
A\times B & & \xrightarrow[M\otimes P\times N\otimes Q]{} & & E\times F
\end{array}
$$

These are invertible when $\mathbb{D}$ is genuinely cartesian.

WHAT ARE THESE COMPARISON CELLS?

Whether $\mathbb{D}$ is just precartesian or genuinely cartesian, $\mathbb{D}_1$ has finite products and

- $\phi_{A,B} = \langle y_{\pi_A}, y_{\pi_B} \rangle$
- $\phi_{(M,N),(P,Q)} = \langle \pi_M \otimes \pi_P, \pi_N \otimes \pi_Q \rangle$

Follows from transformation conditions on unit and conunit of adjunction $\Delta \dashv \times$.

CARTESIAN EQUIPMENT EXAMPLES

- $(\mathscr{V}, \otimes)$ monoidal but additionally cartesian
  1. $\mathbb{A}$**b**
  2. $\mathbb{S}$**lat**
- $\mathbb{S}$**pan** - [Ale18, Prop. 4.2.6]
- $\mathbb{R}$**el** - [Lam22] directly but $\mathbb{R}$**el** $\cong$ $\mathbb{M}$**at**(**2**) so also by below
- $\mathbb{P}$**rof**
- $\mathbb{R}$**ing** - can do this by hand
- $\mathbb{M}$**at**$(\mathscr{V})$ for $\mathscr{V}$ cartesian monoidal [Ale18, Prop 4.2.5]

But what about the others?

And is there a more systematic way to do these proofs?

SOME MAIN PLAYERS IN MORE DETAIL

TWO

Let $\mathbb{2}$ denote the double category with

- one object $\bullet$ and no non-identity arrows
- two proarrows
  1. $0 := y_\bullet$
  2. 1
- one non-identity cell $0 \leq 1$

External composition is $\vee\colon \mathbb{2} \times \mathbb{2} \to \mathbb{2}$.

Cartesian structure is $\wedge\colon \mathbb{2} \times \mathbb{2} \to \mathbb{2}$.

RINGS AND BIMODULES [PAR21]

Let $\mathbb{R}$**ing** denote the double category whose

- objects are unital rings
- morphisms are unital ring homomorphisms
- proarrows are usual bimodules
- cells are homomorphisms of abelian groups $\phi \colon M \to N$

$$\begin{array}{ccc} A & \overset{M}{\longrightarrow} & B \\ {\scriptstyle f}\downarrow & \phi & \downarrow{\scriptstyle g} \\ C & \underset{N}{\longrightarrow} & D \end{array}$$

such that $f(a)\phi(m) = \phi(am)$ and $\phi(m)g(b) = \phi(mb)$.

DOUBLE STRUCTURE OF RINGS AND BIMODULES

- Every ring is a bimodule over itself = external unit
- Tensor of bimodules (where defined) = external composition
- $\mathbb{R}$**ing** has all companions and conjoints = restriction and extension of scalars [Par21, §2]
- One can check by hand that $\mathbb{R}$**ing** is cartesian using Aleiferi's criteria above

ORDERS AND IDEALS [CS86]

Let $\mathbb{I}$**dl** denote the double category whose

- objects are (pre)ordered sets (reflexive and transitive)
- arrows are order-preserving maps (i.e. internal functors)
- proarrows are **ideals** – relations $I \rightarrow A \times B$ such that $a' \leq_A a$ and aIb and $b \leq_B b'$ together imply $a'Ib'$
- cells??? (Carboni & Street define a *bicategory*)

WHAT ARE THE CELLS?

For cells, options:

1. Mimic module definition explicitly
2. Rely on later abstract developments
3. Notice connection to [GP99, §3.3] – namely, the double category of **2**-enriched categories and profunctors

IDEALS ARE PROFUNCTORS

A **2-enriched profunctor** is an order-preserving map $P\colon A^{op} \times B \to \mathbf{2}$.

These are called **pre-order profunctors** in [Gra20, §3.4.6].

Via **2**-*elements construction* such a profunctor yields an ideal:

$$I := \mathbf{Elt}(P) = \{(a, b) \mid P(a, b) = 1\}$$

Check: if $a' \leq_A a$ and aIb and $b \leq_B b'$, then

$$P(a', b') \geq_{\mathbf{2}} P(a, b) \geq_{\mathbf{2}} 1$$

since P contravariant in 1st argument and covariant in 2nd.

Reverse construction just as easy.

CELLS, DEFINED

So, $\mathbb{I}$**dl** is essentially **p**$\mathbb{O}$**rd** from [GP99, §3.3], [Gra20, §3.4.6].

Cells are then certain **2**-natural transformations.

This is an instance of another class of examples, namely, $\mathscr{V}$-$\mathbb{P}$**rof** for suitably structured monoidal $\mathscr{V}$.

1. $\mathscr{V}$ = **2** yields orders and ideals/order-profunctors
2. $\mathscr{V} = \mathbb{R}_+$ yields Lawvere metric space and metric profunctors
3. $\mathscr{V}$ = **Ab** yields preadditive categories and preadditive profunctors
4. $\mathscr{V}$ = **Set** yields ordinary categories and profunctors

In general $\mathscr{V}$ needs a bit of structure:

- monoidal
- closed
- cocomplete
- (if not closed) tensor preserves colimits in each argument

In this case, $\mathscr{V}$-$\mathbb{P}$**rof**

- is a double category (coend formula for external composition)
- is an equipment (we'll see this later)
- if $\mathscr{V}$ is cartesian, so is $\mathscr{V}$-$\mathbb{P}$**rof** (also later).

SUPLATTICES [JT84, CH. I]

A **suplattice** is a poset A for which each arbitrary subset $S \subset A$ has a supremum. A **homomorphism** of suplattices is an order- and join-preserving function. Category denoted **Slat**.

- Every homom f has a right adjoint $f_* y = \bigvee\{x \mid f(x) \leq y\}$
- **Slat** finitely complete and cocomplete
- Tensor $A \otimes B$ makes **Slat** into a monoidal category.
- closed; homs are sets of morphisms $A \to B$ with pointwise order
- strong self-duality
- $*$-autonomy

DOUBLE CATEGORY OF SUPLATTICES

$\mathbb{S}$**lat** is the horizontal categorification of **Slat** with

- one object
- only identity arrows
- a proarrow is a suplattice
- a cell is a suplattice homomorphism.

This is *not* the bicategory of semilattices from [CW87].

Slat is finitely complete and tensor distributes over products [JT84, Prop. I.5.2]. So $\mathbb{S}$**lat** is cartesian. It is an equipment and is closed and has an involution as in [Shu08, §5.8, §10.1].

DOUB CATS OF FRAMES AND OF POINTLESS SPACES

Joyal & Tierney define *frames* and *locales* as certain *monoids* in the monoidal category of suplattices [JT84, Chs. II, III]. Likewise they define modules over such a monoid.

Looking for generalizations:

- $\mathbb{F}\textbf{rame}$ = ???
- $\mathbb{L}\textbf{oc}$ = ???

Since likely $\mathbb{L}\textbf{oc} = \mathbb{F}\textbf{rame}^{op}$ it suffices to define the former.

What are these? Are they cartesian? Equipments?

Pattern:

1. objects are monoid-like
2. proarrows are module-like
3. composition is by certain stable colimits or coends

For this we need monoids and modules in a double category.

MONOIDS AND MODULES

LITERATURE AND TENDENTIOUS OPINIONS

- monoids & modules appear (first?) in Leinster's [Lei04] in context of T-multicategories
- utilized in [CS10] in context of virtual double categories (special T-multicategories)
- appear non-virtually in [Shu08] to show that many double categories are nice equipments
- virtual = correct (double presheaves form a virtual double category [Par11])
- everything is a monoid?

DEFINITIONS

A **monoid** in a double category $\mathbb{D}$ is an endo-proarrow $A\colon X \nrightarrow X$ together with globular action $\mu\colon A \otimes A \Rightarrow A$ and unit $\eta\colon\ y_X \Rightarrow A$ cells satisfying the equations

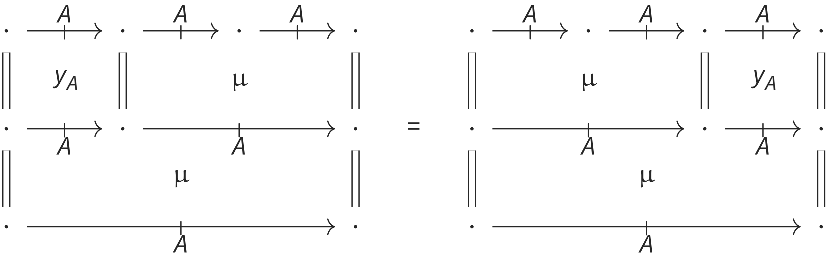
1. $\mu(\mu \otimes\ y_A) = \mu(y_A \otimes \mu)$ $\qquad a(a'a'') = (aa')a''$
2. $\mu(\eta \otimes\ y_A) =\ y_A = \mu(y_A \otimes \eta).$ $\qquad 1a = a = a1$

A **homomorphism** of such monoids consists of an arrow $f\colon X \to Y$ and a cell $\phi\colon A \Rightarrow B$ with source and target f satisfying

1. $\nu(\phi \otimes \phi) = \phi\mu$ $\qquad \phi(a) \cdot \phi(a') = \phi(a \cdot a')$
2. $\phi\eta = \epsilon\, y_f.$ $\qquad \phi(1) = 1$

Let **Mon**($\mathbb{D}$) be the category of monoids and homomorphisms in $\mathbb{D}$.

ASSOCIATIVITY CONDITION DIAGRAMATICALLY



EXAMPLES

- A monoid in $\mathbb{S}$**pan** is a small category.
- A monoid in $\mathbb{R}$**el** is an ordered set.
- A monoid in $\mathbb{A}$**b** is a ring with 1.
- A monoid in $\mathbb{M}$**at**$(\mathscr{V})$ is a $\mathscr{V}$-category:
  1. orders
  2. Lawvere metric spaces
  3. preadditive categories
- A monoid A in $\mathbb{S}$**lat** satisfying $a \leq 1$ and $a \cdot a = a$ for all $a \in A$ is a frame, and conversely [JT84, §III.1].

DEFINITIONS

A **bimodule** from a monoid A to one B consists of a proarrow $M\colon X \nrightarrow Y$ and left $\lambda\colon A \otimes M \Rightarrow M$ and right $\rho\colon M \otimes B \Rightarrow M$ globular action cells satisfying

1. $\lambda(y_A \otimes \lambda) = \lambda(\mu \otimes y_M)$ $\qquad a \cdot (a' \cdot m) = (aa') \cdot m$
2. $\rho(\rho \otimes y_B) = \rho(y_M \otimes \nu)$ $\qquad (m \cdot b) \cdot b' = m \cdot (bb')$
3. $\rho(\lambda \otimes y_B) = \lambda(y_A \otimes \rho)$. $\qquad (a \cdot m) \cdot b = a \cdot (m \cdot b)$

A **modulation** between bimodules M and N is a cell $\theta\colon M \Rightarrow N$ where

1. $\lambda(\phi \otimes \theta) = \theta\lambda$ $\qquad \phi(a) \cdot \theta(m) = \theta(a \cdot m)$
2. $\rho(\theta \otimes \psi) = \theta\rho$ $\qquad \theta(m) \cdot \psi(b) = \theta(m \cdot b)$

DEFINITIONS

A **bimodule** from a monoid A to one B consists of a proarrow $M\colon X \nrightarrow Y$ and left $\lambda\colon A \otimes M \Rightarrow M$ and right $\rho\colon M \otimes B \Rightarrow M$ globular action cells satisfying

1. $\lambda(y_A \otimes \lambda) = \lambda(\mu \otimes y_M)$ $\quad a \cdot (a' \cdot m) = (aa') \cdot m$
2. $\rho(\rho \otimes y_B) = \rho(y_M \otimes \nu)$ $\quad (m \cdot b) \cdot b' = m \cdot (bb')$
3. $\rho(\lambda \otimes y_B) = \lambda(y_A \otimes \rho)$. $\quad (a \cdot m) \cdot b = a \cdot (m \cdot b)$

A **modulation** between bimodules M and N is a cell $\theta\colon M \Rightarrow N$ where

1. $\lambda(\phi \otimes \theta) = \theta\lambda$ $\quad \phi(a) \cdot \theta(m) = \theta(a \cdot m)$
2. $\rho(\theta \otimes \psi) = \theta\rho$ $\quad \theta(m) \cdot \psi(b) = \theta(m \cdot b)$

EXAMPLES

- a bimodule in $\mathbb{R}$**ing** is a usual bimodule; a modulation is a properly bilinear map as in [Par21]
- generally a bimodule in $\mathbb{M}$**at**($\mathscr{V}$) is a $\mathscr{V}$-profunctor

PROPOSITION

Monoids, modules, homomorphisms, and certain multicells in a double category $\mathbb{D}$ form a *virtual* double category $\mathbb{M}\mathbf{od}(\mathbb{D})$.

A *virtual double category* is a category equipped proarrows and further cells with multi-sources. Cells compose like in operads or multicategories, but not externally as in a double category.

DEFINITION AND PROP [SHU08, §11.4, §11.10]

A double category $\mathbb{D}$ has **local coequalizers** if each category $\mathbb{D}(A, B)$ has coequalizers and they are preserved by external composition in each argument.

If $\mathbb{D}$ is an equipment with local coequalizers, then $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a double category and in fact an equipment.

EXAMPLES

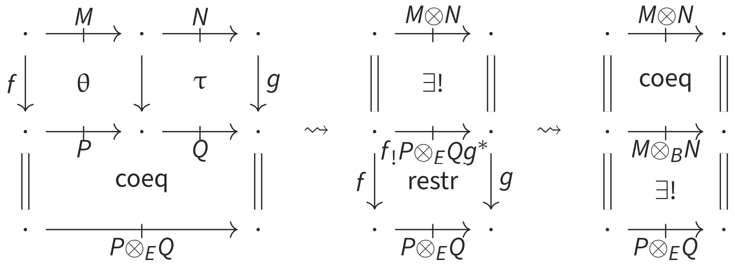
- $\mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathscr{V}))$:= $\mathscr{V}$-$\mathbb{P}\mathbf{rof}$ - $\mathscr{V}$-categories and $\mathscr{V}$-profunctors
  1. $\mathbb{M}\mathbf{et} \cong \mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathbb{R}_+))$
  2. $\mathbb{I}\mathbf{dl} \cong \mathbb{M}\mathbf{od}(\mathbb{R}\mathbf{el}) \cong \mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathbf{2}))$
  3. $\mathbb{R}\mathbf{ing} \cong \mathbb{M}\mathbf{od}(\mathbb{A}\mathbf{b})$
- $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathscr{E}))$ for finitely complete $\mathscr{E}$ - internal categories, internal functors, internal profunctors
  1. $\mathbb{P}\mathbf{rof} \cong \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}) \cong \mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathbf{Set}))$
  2. $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{Esp}))$ or $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{CGHaus}))$ - double spaces??
  3. $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{Man}))$ - double manifolds?? (virtual!)
- $\mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{od}(\mathbb{D}))$ - algebras and algebra bimodules

COMMENTS ON THE PROOF

Composition of modules defined via a coequalizer in $\mathbb{D}(A, C)$:

$$M \otimes B \otimes N \rightrightarrows M \otimes N \to M \otimes_B N$$

Then for modulations $\theta\colon M \Rightarrow P$ and $\tau\colon N \Rightarrow Q$,



THEOREM

If $\mathbb{D}$ is a cartesian equipment with local coequalizers, then $\mathbb{M}\mathbf{od}(\mathbb{D})$ is a cartesian equipment.

NB: there's almost certainly a finer analysis to be done here.
You can see some further proof details at
https://michaeljlambert.github.io/draft(6June2023).pdf

PROOF COMMENTS

Use Aleiferi's criteria:

1. is an equipment ✓
2. 0-part has products
3. has local products
4. laxators are invertible ✓

Checkmarks: [Shu08, Prop. 11.10] proves the first one; last one: laxators are induced from those of $\mathbb{D}$.

PROOF COMMENTS

If (X, A, μ, η) and (Y, B, ν, ϵ) are monoids in $\mathbb{D}$, then the product in $\mathbb{D}_1$, namely, $A \times B \colon X \times Y \nrightarrow X \times Y$ has induced monoid structure.

Unit and multiplication:

$$
\begin{array}{ccc}
\cdot & \xrightarrow[\quad]{y_{X\times Y}} & \cdot \\
\| & \text{can}\cong & \| \\
\cdot & \xrightarrow[y_X \times y_Y]{\quad} & \cdot \\
\| & \eta\times\epsilon & \| \\
\cdot & \xrightarrow[A\times B]{\quad} & \cdot
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\cdot & \xrightarrow{A\times B} & \cdot & \xrightarrow{A\times B} & \cdot \\
\| & & \text{can}\cong & & \| \\
\cdot & & \xrightarrow[A\otimes A\times B\otimes B]{\qquad\qquad} & & \cdot \\
\| & & \mu\times\nu & & \| \\
\cdot & & \xrightarrow[A\times B]{\qquad\qquad} & & \cdot
\end{array}
$$

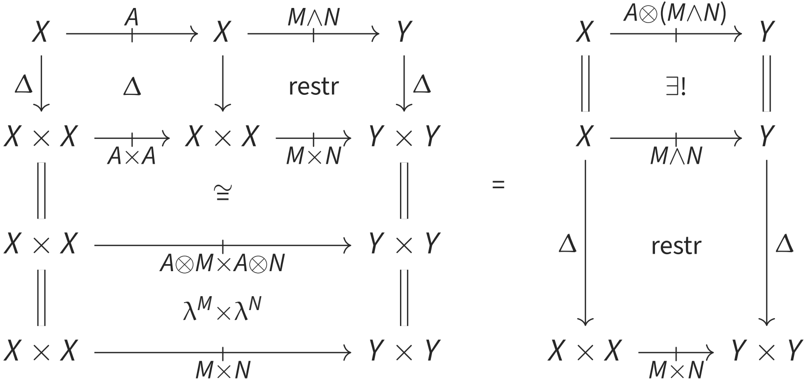
The can isos are the laxity cells given by the pairing of projections.

PROOF COMMENTS

- canonical projections coequalize each side of the two required monoid equations
- consequently, these equations hold by uniqueness (that is, $A \times B$ is a monoid)
- universality by showing *given* projection and pairing morphisms in $\mathbb{D}_1$ are monoid homomorphisms (again by a uniqueness argument via projections)

PROOF COMMENTS

For bimodules M and N between monoids $A \nrightarrow B$, the local product $M \wedge N$ in $\mathbb{D}_1$ is a bimodule. Left action induced:



Likewise for the right action.

PROOF COMMENTS

- again projections coequalize some required equations, but need to take account of restrictions too!
- so, uniqueness applied 2x to get the bimodule equations
- again *given* projections and pairing morphisms for local products in $\mathbb{D}$ are modulations
- see this using uniqueness arguments once again

EXAMPLES

The following are thus all cartesian equipments:

- $\mathbb{P}\mathbf{rof} \cong \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}) \cong \mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathbf{Set}))$
- $\mathbb{I}\mathbf{dl} \cong \mathbb{M}\mathbf{od}(\mathbb{R}\mathbf{el}) \cong \mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{at}(\mathbf{2}))$
- $\mathbb{R}\mathbf{ing} \cong \mathbb{M}\mathbf{od}(\mathbb{A}\mathbf{b})$
- $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{lat})$ - monoids and modules in semilattices
- $\mathbb{D}\mathbf{Esp} := \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{Esp}))$ - double category of *double spaces*

DOUBLE CATEGORIES OF SPACES

Have sub-double category $\mathbb{F}\mathbf{rame} \hookrightarrow \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{lat})$ of frames, homomorphisms and modules generalizing [JT84].

Likewise $\mathbb{L}\mathbf{oc} := \mathbb{F}\mathbf{rame}^{op}$.

To do:

1. A cell-theoretic definition of a frame
2. Stone Duality? That is, is $\mathbb{S}\mathbf{pan}(\mathbf{Esp})$ the *right* double category of spaces for a Stone-type duality

$$\mathscr{O} : \mathbb{S}\mathbf{pan}(\mathbf{Esp}) \rightleftarrows \mathbb{L}\mathbf{oc} : \mathrm{pt}$$

3. Descent theory for modules phrase purely double-theoretically?

REFERENCES |

[Ale18] Evangelia Aleiferi.
Cartesian Double Categories with an Emphasis on Characterizing Spans.
PhD thesis, Dalhousie University, 2018.

[CKWW08] A. Carboni, G.M. Kelly, R.F.C. Walters, and R.J. Wood.
Cartesian bicategories II.
Theory and Applications of Categories, 19(6):93–124, 2008.

[CS86] Aurelio Carboni and Ross Street.
Order ideals in categories.
Pacific Journal of Mathematics, 124(2):275–288, 1986.

[II

references]references
[CS10] G.S.H. Cruttwell and M.A. Shulman.
A unified framework for generalized multicategories.
Theory and Applications of Categories, 24:580–655, 2010.

[CW87] A. Carboni and R.C.F. Walters.
Cartesian bicategories I.
Journal of Pure and Applied Algebra, 49:11–32, 1987.

[DPP10] Robert Dawson, Robert Paré, and Dorette Pronk.
The span construction.
Theory and Applications of Categories, 24(13):302–377, 2010.

[][

[GP99] Marco Grandis and Robert Paré.
Limits in double categories.
Cahiers de topologie et géométrie différentielle catégoriques, 40(3):162–220, 1999.

[GP04] Marco Grandis and Robert Paré.
Adjoints for double categories.
Cahiers de Topologie et Géométrie Différentielle Catégoriques, 45(3):193–240, 2004.

[][
[Gra20] Marco Grandis.
Higher Dimensional Categories: From Double to Multiple Categories.
World Scientific, 2020.

[JT84] André Joyal and Myles Tierney.
An extension of the galois theory of grothendieck.
Memoirs of the American Mathematical Society, 51(309), 1984.

[V

[][
[Lam22] Michael Lambert.
Double categories of relations.
Theory and Applications of Categories, 38(33):1249–1283, 2022.

[Lei04] Tom Leinster.
Higher Operads, Higher Categories, volume 298 of *London Mathematical Society Lecture Notes Series*.
Cambridge University Press, 2004.

[VI

[][

[Par11] Robert Paré.
Yoneda theory for double categories.
Theory and Applications of Categories, 25(17):436–489, 2011.

[Par21] Robert Paré.
Morphisms of rings.
In Claudia Casadio and Philip J. Scott, editors, *Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics*, pages 271–298. Springer Verlag, 2021.

[][
[Shu08] Michael Shulman.
Framed bicategories and monoidal fibrations.
Theory and Applications of Categories, 20(18):650–738, 2008.

[Woo82] R. Wood.
Abstract pro arrows I.
Cahiers de Topologie et Géom. Diff. Catégoriques, 23:279–290, 1982.