Classical Distributive Restriction Categories
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JS PL (he/him), joint work with Robin Cockett

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Email: `js.lemay@mq.edu.au`

Website: `https://sites.google.com/view/jspl-personal-webpage`

Mt.Fuji Adventure

Start of hike.

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Top of Mt.Fuji.
(Fun fact: this was on my bday)

Mt.Fuji Adventure

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End of hike.

Products in PAR

In the category of sets and partial functions, PAR: ($\uparrow$ means undefined and $\downarrow$ means defined)

- The disjoint union $\sqcup$ is still the coproduct.
- The Cartesian product $\times$ is not the product! Here's a reason why:

$$
\begin{tikzcd}
 & \{*\} \arrow[dl, "\uparrow"'] \arrow[d] \arrow[dr, equal] & \\
\varnothing & \times\{*\} \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & \{*\}
\end{tikzcd}
$$

No map $\{*\} \to \varnothing \times \{*\}$ exists that makes this diagram commute!

Products in PAR

In the category of sets and partial functions, PAR: ($\uparrow$ means undefined and $\downarrow$ means defined)

- The disjoint union $\sqcup$ is still the coproduct.
- The Cartesian product $\times$ is not the product!
- However, PAR still has products given by: $X \sqcup Y \sqcup (X \times Y)$, where the projections are $p_0 : X \sqcup Y \sqcup (X \times Y) \to X$ and $p_1 : X \sqcup Y \sqcup (X \times Y) \to Y$ are defined as:

$$
\begin{array}{ccc}
p_0(x) = x & p_0(y) \uparrow & p_0(x, y) = x \\
p_1(x) \uparrow & p_1(y) = y & p_1(x, y) = y
\end{array}
$$

The pairing is defined as follows:

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "\exists!\ \langle\langle f, g \rangle\rangle"'] \arrow[dr, "g"] & \\
X & X \sqcup Y \sqcup (X \times Y) \arrow[l, "p_0"] \arrow[r, "p_1"'] & X
\end{tikzcd}
$$

$$
\langle\langle f, g \rangle\rangle(z) = \begin{cases} f(z) & \text{if } f(z) \downarrow \text{ and } g(z) \uparrow \\ g(z) & \text{if } f(z) \uparrow \text{ and } g(z) \downarrow \\ (f(z), g(z)) & \text{if } f(z) \downarrow \text{ and } g(z) \downarrow \\ \uparrow & \text{if } f(z) \uparrow \text{ and } g(z) \uparrow \end{cases}
$$

Today's Story

- Give a **restriction category** explanation of what's going.

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- Give a **restriction category** explanation of what's going.
- We will explain when in a **distributive** restriction category, $A \oplus B \oplus (A \times B)$ is a product.

Theorem (Cockett and Lemay)

For a distributive restriction category $\mathbb{X}$, the following are equivalent:

1. $A \oplus B \oplus (A \times B)$ *is a product;*
2. $\mathbb{X}$ *is* **classical***;*
3. $\mathbb{X}$ *is the Kleisli category of the exception monad* $\_ \oplus 1$ *of a distributive category.*

Restriction Categories

Definition

A **restriction category** is a category $\mathbb{X}$ equipped with a **restriction operator** $\overline{(\ \ )}$, which associates every map $f : A \to B$ to a map $\overline{f} : A \to A$, called the **restriction of** f, and such that the following four axioms hold[a]:

- **[R.1]** $\overline{f}f = f$
- **[R.2]** $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$
- **[R.3]** $\overline{\overline{g}f} = \overline{g}\,\overline{f}$
- **[R.4]** $f\overline{g} = \overline{fg}f$

A map f is total if $\overline{f} = 1$.

Cockett, R. and Lack, S. **Restriction Categories I: Categories of partial maps.**.

[a] Composition written in diagramatic order

Restriction Categories – Examples

Example

PAR is a restriction category where the restriction of a partial function $f : X \to Y$ is the partial function $\overline{f} : X \to X$ defined as follows:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x)\downarrow \\ \uparrow & \text{if } f(x)\uparrow \end{cases}$$

Total maps are precisely ordinary total functions between sets.

Example

Let k be a commutative ring and let $k\text{-CALG}_\bullet$ be the category whose objects are commutative k-algebras and whose maps are *non-unital* k-algebra morphisms, that is, k-linear morphisms $f : A \to B$ that preserve the multiplication, $f(ab) = f(a)f(b)$, but not necessarily the multiplicative unit, so $f(1)$ may not equal 1.

$k\text{-CALG}_\bullet^{op}$ is a restriction category, so $k\text{-CALG}_\bullet$ is a corestriction category where the corestriction of a non-unital k-algebra morphism $f : A \to B$ is the non-unital k-algebra morphism $\overline{f} : B \to B$ defined as:

$$\overline{f}(b) = f(1)b$$

Restriction Products

Definition

A **Cartesian restriction category** is a restriction category $\mathbb{X}$ with:

- A **restriction terminal object**, that is, an object 1 such that for every object A there exists a unique total map $t_A : A \to 1$ such that for every map $f : A \to B$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\overline{f}"'] & B \arrow[d, "t_B"] \\ A \arrow[r, "t_A"'] & 1 \end{tikzcd} \tag{1}$$

- Binary **restriction products**, that is, every pair of objects A and B, there is an object $A \times B$ with total maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that for every pair of maps $f : C \to A$ and $g : C \to B$, there exists a unique map $\langle f, g \rangle : C \to A \times B$ such that:

$$\begin{tikzcd} C \arrow[d, "f"'] & C \arrow[l, "\overline{g}"'] \arrow[r, "\overline{f}"] \arrow[d, dashed, "\exists ! \ \langle f, g \rangle"] & C \arrow[d, "g"] \\ A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B \end{tikzcd} \tag{2}$$

Restriction Product – Examples

Example

PAR has restriction products:

- The restriction terminal object is a chosen singleton $1 = \{*\}$, and $t_X : X \to \{*\}$ maps everything to the single element, $t_X(x) = *$.
- The restriction product is given by the Cartesian product $A \times B$, where the projections $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ are defined as $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$, and the pairing of partial functions is defined as:

$$\langle f, g\rangle(x) = \begin{cases} (f(x), g(x)) & \text{if } f(x)\downarrow \text{ and } g(x)\downarrow \\ \uparrow & \text{o.w.} \end{cases}$$

Example

k-$\mathsf{CALG}_\bullet^{op}$ has restriction products, so k-$\mathsf{CALG}_\bullet$ has corestriction coproducts where:

- The corestriction initial object is k, and $t_A : k \to A$ is defined by the k-algebra structure of A.
- The corestriction coproduct is given by the tensor product $A \otimes B$, where the injections $\iota_0 : A \to A \otimes B$ and $\iota_1 : B \to A \otimes B$ are defined as $\iota_0(a) = a \otimes 1$ and $\iota_1(b) = 1 \otimes b$, and where the copairing is given by:

$$[f, g](a \otimes b) = f(a)g(b)$$

Restriction Coproducts

- Restriction coproducts are actual coproducts in the usual sense.
- While we will only need to work with binary restriction products, for simplicity, it will be easier to work with finite (restriction) coproducts.
- For a category $\mathbb{X}$ with finite coproducts, we denote the coproduct as $\oplus$, with injection maps $\iota_j : A_j \to A_0 \oplus \cdots \oplus A_n$, where the copairing operation is denoted by $[-, \cdots, -]$, and we denote the initial object as 0 with unique map $z_A : 0 \to A$.

Definition

A **coCartesian restriction category** is a restriction category $\mathbb{X}$ with finite **restriction coproducts**, that is, $\mathbb{X}$ has finite coproducts where all the injection maps $\iota_j : A_j \to A_0 \oplus \cdots \oplus A_n$ are total.

Restriction Coproduct – Examples

Example

PAR has restriction coproducts, the initial object is the empty set $0 = \varnothing$ and the coproduct is given by disjoint union $X \oplus Y = X \sqcup Y$.

Example

k-CALG$_\bullet^{op}$ has restriction coproducts, so k-CALG$_\bullet$ has corestriction products where the terminal object is the zero algebra 0 and the product is given by the product of k-algebras $A \times B$.

Distributive Restriction Categories

Definition

A **distributive restriction category** is a restriction category $\mathbb{X}$ which is both a Cartesian restriction category and a coCartesian restriction category such that:

$$(A \times B) \oplus (A \times C) \cong A \times (B \oplus C) \qquad\qquad 0 \cong A \times 0 \tag{3}$$

Cockett, R. and Lack, S. **Restriction Categories III: Colimits, Partial Limits and Extensivity**.

Example

PAR is a distributive restriction category.

Example

k-CALG$_\bullet^{op}$ is a distributive restriction category.

Back to today's question:

- So now we want to ask when in a distributive category is $A \oplus B \oplus (A \times B)$ a product...

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- So now we want to ask when in a distributive category is $A \oplus B \oplus (A \times B)$ a product...
- But first we need to define the projections! To do that we need restriction zeroes...

Restriction Zeroes

Definition

A restriction category $\mathbb{X}$ is said to have **restriction zeroes** if $\mathbb{X}$ has zero maps such that $\overline{0} = 0$.

Example

PAR has restriction zeroes, where the restriction zero maps are the partial functions $0 : X \to Y$ which is nowhere defined, $0(x) \uparrow$ for all $x \in X$.

Example

k-$\mathrm{CALG}_\bullet^{op}$ has restriction zeroes, so k-$\mathrm{CALG}_\bullet$ has corestriction zeroes, where $0 : A \to B$ is the zero morphism, $0(a) = 0$.

Classical Products

Definition (Cockett and Lemay)

A distributive restriction category $\mathbb{X}$ is said to have **classical products** if $\mathbb{X}$ has restriction zeroes and for every pair of objects A and B, the following is a product diagram:

$$A \xleftarrow{p_0 := [1_A, 0, \pi_0]} A \oplus B \oplus (A \times B) \xrightarrow{p_1 := [0, 1_B, \pi_1]} B \tag{4}$$

We call:

$$A \& B := A \oplus B \oplus (A \times B)$$

the **classical product** and $p_0 : A\&B \to A$ and $p_1 : A\&B \to B$ the **classical projections**.

Lemma

A distributive restriction category with classical products has finite products (the terminal object is the restriction initial object 0*).*

Classical Products – Examples

Example

PAR has classical products where:

$$X \& Y = X \sqcup Y \sqcup (X \times Y)$$

Example

k-$\mathsf{CALG}_\bullet^{op}$ has classical products, so k-$\mathsf{CALG}_\bullet$ has coclassical coproducts where:

$$A \& B = A \times B \times (A \otimes B)$$

and the coclassical injections $p_0 : A \to A \& B$ and $p_1 : B \to A \& B$ are defined as:

$$p_0(a) = (a, 0, a \otimes 1) \qquad\qquad p_1(b) = (0, b, 1 \otimes b)$$

The coclassical copairing $\langle\!\langle f, g \rangle\!\rangle : A \& B \to C$ is defined as follows:

$$\langle\!\langle f, g \rangle\!\rangle(a, b, x \otimes y) = f(a) - f(a)g(1) + g(b) - f(1)g(b) + f(x)g(y)$$

Classical Products

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$$A \xleftarrow{p_0 := [1_A, 0, \pi_0]} A \oplus B \oplus (A \times B) \xrightarrow{p_1 := [0, 1_B, \pi_1]} B \tag{5}$$

We call:

$$A \& B := A \oplus B \oplus (A \times B)$$

the **classical product** and $p_0 : A\&B \to A$ and $p_1 : A\&B \to B$ the **classical projections**.

Why the name **classical** products?

Because a distributive restriction category has **classical products if and only if it classical!**

Classical Restriction Categories

A classical restriction category is a restriction category with:

- Joins
- Relative Complements

This allows for classical Boolean reasoning.

Joins

In a restriction category for parallel maps $f : A \to B$ and $g : A \to B$ we say that:

- f is **less than or equal to** g, $f \leq g$, if $\overline{f}g = f$;
  IDEA: When $f(x)$ is defined then $g(x)$ is defined and $g(x) = f(x)$
- f and g are **compatible**, $f \smile g$, if $\overline{f}g = \overline{g}f$
  IDEA: When both $f(x)$ and $g(x)$ are defined then $f(x) = g(x)$

Joins

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Definition

A **join restriction category** is a restriction category $\mathbb{X}$ such that for any finite family of parallel maps $f_0 : A \to B$, ..., $f_n : A \to B$ that is pairwise compatible, so $f_i \smile f_j$ for all $0 \leq i, j, \leq n$, is a (necessarily unique) map $f_0 \vee \cdots \vee f_n : A \to B$ such that:

- $f_i \leq f_0 \vee \cdots \vee f_n$ for all $0 \leq i \leq n$;
- If $g : A \to B$ is a map such that $f_i \leq g$ for all $0 \leq i \leq n$ then $f_0 \vee \cdots \vee f_n \leq g$;
- $h(f_0 \vee \cdots \vee f_n) = hf_0 \vee \cdots \vee hf_n$.

Join restriction categories have restriction zeroes given by the join of the empty family.

Example

PAR is a join restriction category where

- $f \leq g$ if $g(x) = f(x)$ whenever $f(x)\downarrow$.
- $f \smile g$ if $f(x) = g(x)$ whenever both $f(x)\downarrow$ and $g(x)\downarrow$.
- If $f \smile g$, then their join $f \vee g$ is defined as follows:

$$(f \vee g)(x) = \begin{cases} f(x) & \text{if } f(x)\downarrow \text{ and } g(x)\uparrow \\ g(x) & \text{if } f(x)\uparrow \text{ and } g(x)\downarrow \\ f(x) = g(x) & \text{if } f(x)\downarrow \text{ and } g(x)\downarrow \\ \uparrow & \text{if } f(x)\uparrow \text{ and } g(x)\uparrow \end{cases}$$

Example

k-$\mathrm{CALG}_\bullet^{op}$ is a join restriction category, so k-$\mathrm{CALG}_\bullet$ is a (co?)join corestriction category:

- $f \leq g$ if $f(1)g(a) = f(a)$.
- $f \smile g$ if $g(1)f(a) = f(1)g(a)$.
- If $f \smile g$, then their join $f \vee g$ is defined as follows:

$$(f \vee g)(a) = f(a) + g(a) + g(1)f(a) = f(a) + g(a) + g(1)f(a)$$

Classical Restriction Categories

In a restriction category with restriction categories for parallel maps $f : A \to B$ and $g : A \to B$ we say that:

- f and g are **disjoint**, $f \perp g$, if $\overline{f} g = 0$ (or equivalently $\overline{g} f = 0$).

  IDEA: When $f(x)$ is defined then $g(x)$ is undefined, and vice versa.

  NOTE: $f \perp g \Rightarrow f \smile g$

Classical Restriction Categories

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  NOTE: $f \perp g \Rightarrow f \smile g$

Definition

A **classical restriction category** is a join restriction category $\mathbb{X}$ such that all for parallel maps $f : A \to B$ and $g : A \to B$ such that $f \leq g$, there exists a (necessarily unique) map $g \backslash f : A \to B$, called the **relative complement**, such that:

- $g \backslash f \perp f$
- $g \backslash f \vee f = g$

Cockett, R. and Manes, E. **Boolean and Classical Restriction Categories.**.

Classical Restriction Categories – Examples

Example

PAR is a classical restriction category where if $f \leq g$, then the relative complement f/g is defined as follows:

$$(g\backslash f)(x) = \begin{cases} g(x) & \text{if } f(x)\uparrow \text{ and } g(x)\downarrow \\ \uparrow & \text{if } f(x)\downarrow \text{ or } g(x)\uparrow \end{cases}$$

Example

k-$\mathrm{CALG}_\bullet^{op}$ is a classical restriction category, so k-$\mathrm{CALG}_\bullet$ is a coclassical corestriction category where if $f \leq g$, then the relative complement $g\backslash f$ is defined as follows:

$$(g\backslash f)(a) = g(a) - f(1)g(a)$$

Main Result

Theorem (Cockett and Lemay)

A distributive restriction category is classical if and only if it has classical products.

From Classical to Classical Products

Define the classical pairing $\langle\!\langle f, g \rangle\!\rangle : C \to A \oplus B \oplus (A \times B)$ as the join of these three maps:

$$\langle\!\langle f, g \rangle\!\rangle := (f \backslash (\overline{g} f))\, \iota_0 \vee \big(g \backslash (\overline{f} g)\big)\, \iota_1 \vee \langle f, g \rangle \iota_2 \tag{6}$$

From Classical Products to Classical

If $f \smile g$ then define their join $f \vee g$ as follows:

$$f \vee g := \; A \xrightarrow{\langle\!\langle f,g \rangle\!\rangle} B \oplus B \oplus (B \times B) \xrightarrow{[1_B, 1_B, \pi_0]} B \tag{7}$$

If $f \leq g$ then define their relative complement $g \backslash f$ as follows:

$$g \backslash f := \; A \xrightarrow{\langle\!\langle f,g \rangle\!\rangle} B \oplus B \oplus (B \times B) \xrightarrow{[0, 1_B, 0]} B \tag{8}$$

Exception Monads

- It turns out that classical distributive restriction categories are precisely the Kleisli categories of exception monads!

Exception Monads

A distributive category is a category $\mathbb{D}$ with finite products ($\times$ and 1) and finite coproducts ($\oplus$ and 0) such that:

$$(A \times B) \oplus (A \times C) \cong A \times (B \oplus C) \qquad\qquad 0 \cong A \times 0 \tag{9}$$

The **exception monad** of $\mathbb{D}$ is the monad defined as $\_ \oplus 1$.

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The **exception monad** of $\mathbb{D}$ is the monad defined as $\_ \oplus 1$.

Proposition (Cockett and Lack)

The Kleisli category $\mathbb{D}_{\_\oplus 1}$ is a distributive restriction category where:

- *For a Kleisli map $f : A \to B \oplus 1$, its restriction is:*

$$A \xrightarrow{\langle 1_A, f \rangle} A \times (B \oplus 1) \cong (A \times B) \oplus (A \times 1) \xrightarrow{\pi_0 \oplus \pi_1} A \oplus 1 \tag{10}$$

- *$\times$ is the restriction product and 1 is the restriction terminal object;*
- *$\oplus$ is the restriction coproduct and 0 is the restriction initial object.*

Lemma

The Kleisli category $\mathbb{D}_{\_\oplus 1}$ has products given by $A \oplus B \oplus (A \times B)$.

This is because $(A \times B) \oplus 1 \cong (A \oplus B \oplus (A \times B)) \oplus 1$.

Proposition (Cockett and Lemay)

The Kleisli category $\mathbb{D}_{\_\oplus 1}$ is a classical distributive restriction category.

For any distributive restriction category $\mathbb{X}$, its subcategory of total maps $T[\mathbb{X}]$ is a distributive category.

Proposition (Cockett and Lemay)

A distributive restriction category $\mathbb{X}$ is classical/has classical products if and only if $\mathbb{X} \cong T[\mathbb{X}]_{\_\oplus 1}$.

Given a total map $f : A \to B \oplus 1$, define $f^\sharp : A \to B$ as the composite:

$$f^\sharp := A \xrightarrow{f} B \oplus 1 \xrightarrow{[1_B,0]} B \tag{11}$$

Given a map $g : A \to B$, define the total map $g^\flat : A \to B \oplus 1$ as the join:

$$g^\flat := g\iota_0 \vee t_A \backslash (\overline{g} t_A)\iota_1$$

IDEA: Where g is defined, send it to B and where g is undefined sent it to 1.

Main Result Again

Theorem (Cockett and Lemay)

For a distributive restriction category $\mathbb{X}$, the following are equivalent:

1. $\mathbb{X}$ *is classical;*
2. $\mathbb{X}$ *has classical products;*
3. *There is a distributive category* $\mathbb{D}$ *such that* $\mathbb{X}$ *is restriction equivalent to* $\mathbb{D}_{\_\oplus 1}$.

A Wise Man Climbs Mt.Fuji Once, Only A Fool Would Climb it Twice

By this logic, Robin is a wise man....

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By this logic, Robin is a wise man.... I'm a certified fool (baka)!

Aug 15 2022

Aug 30 2022

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HOPE YOU ENJOYED MY TALK!

THANK YOU FOR LISTENING!

MERCI!

Email: `js.lemay@mq.edu.au`

Website: `https://sites.google.com/view/jspl-personal-webpage`

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