Partial Monoids

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Why I care

- Monomials (nonzero products of generators) in $\frac{\mathbb{N}[x_1,\dots,x_n]}{I}$
- For example $\frac{\mathbb{N}[x,y,z]}{\langle x^2,y^2,xz\rangle}$ has monomials $1, x, y, z, xy, yz, z^2, z^3, \dots$
- There

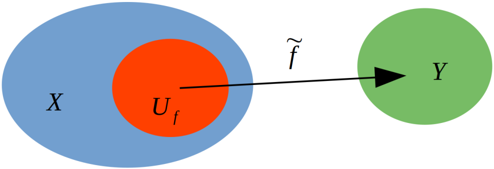
$$x \cdot y = xy \qquad y \cdot z = yz \qquad z \cdot z = z^2$$

- but $x \cdot z = 0$ and $x \cdot x = 0$ are not monomials and thus not defined.

$\Rightarrow$ partial multiplication $\mu : X \times X \to X$ needed

Partial maps in Set

A partial map of sets $f : X \to Y$ is a subset $U_f \subset X$ together with a map $\tilde{f} : U_f \to Y$.



Notations

For maps $f, g : A \to B$, $\mathbf{f} \leq \mathbf{g}$ if $U_f \subset U_g$ and $\tilde{g}|_{U_f} = f$

The **restriction** $\bar{f} : X \to X$ is U_f with the inclusion $U_f \hookrightarrow X$

Restriction categories

Definition

A **restriction category** is a category with a structure that assigns to each map $f : A \to B$ a map $\bar{f} : A \to A$ such that

(R.1) $\bar{f} f = f$

(R.2) $\bar{g}\bar{f} = \bar{f}\bar{g}$

(R.3) $\overline{\bar{g} f} = \bar{g}\bar{f}$

(R.4) $f\bar{h} = \overline{fh} f$

Notation
A map f is **total** if $\bar{f} = 1_A$.
For maps $f, g : A \to B$, $f \leq g$ if $\bar{f} g = f$
This partial order on $\mathrm{Hom}(A, B)$ makes a bicategory.

Partial commutative monoids

Definition

A partial commutative monoid in a restriction category $\mathbb{X}$ (with a restriction terminal object) is an object X with

- all restriction product powers $X \times ... \times X$ of X existing,
- a partial map $m : X \times X \to X$, and
- a total map $u : * \to X$

such that the following diagrams commute:

$$
\begin{tikzcd}
X \arrow[rr, "{(u, 1_X)}"] \arrow[dr, "1_X"'] & \arrow[d, phantom, "="] & X \times X \arrow[dl, "m"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times X \arrow[rr, "{(\mathrm{pr}_1, \mathrm{pr}_0)}"] \arrow[dr, "m"'] & \arrow[d, phantom, "="] & X \times X \arrow[dl, "m"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times X \times X \arrow[r, "m \times 1_X"] \arrow[d, "1_X \times m"'] \arrow[dr, phantom, "="] & X \times X \arrow[d, "m"] \\
X \times X \arrow[r, "m"'] & X
\end{tikzcd}
$$

Morphisms

A **strict morphism of partial monoids** $(X, m, u) \to (X', m', u')$ is a total morphism $f : X \to X'$ such that the following diagrams commute:

$$\begin{tikzcd}
X \times X \arrow[r, "f \times f"] \arrow[d, "m"'] \arrow[dr, phantom, "= / \leq"] & X' \times X' \arrow[d, "m'"] \\
X \arrow[r, "f"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & * \arrow[dl, "u"'] \arrow[dr, "u'"] \arrow[d, phantom, "="] & \\
X \arrow[rr, "f"'] & & X'
\end{tikzcd}$$

Morphisms

A **strict morphism of partial monoids** $(X, m, u) \to (X', m', u')$ is a total morphism $f : X \to X'$ such that the following diagrams commute:

$$
\begin{tikzcd}
X \times X \arrow[r, "f \times f"] \arrow[d, "m"'] \arrow[dr, phantom, "= / \textcolor{blue}{\leq}"] & X' \times X' \arrow[d, "m'"] \\
X \arrow[r, "f"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
& * \arrow[dl, "u"'] \arrow[dr, "u'"] \arrow[d, phantom, "="] & \\
X \arrow[rr, "f"'] & & X'
\end{tikzcd}
$$

A **lax morphism of partial monoids** $(X, m, u) \to (X', m', u')$ is a total morphism $f : X \to X'$ such that the diagrams above commute up to $\leq$:

$$mf \leq (f \times f)m$$

We obtain two categories **SPCM($\mathbb{X}$)** and **LPCM($\mathbb{X}$)**.

Example

$X = \{1, x\}$ with $u(*) = 1, m(1,1) = 1, m(1,x) = x$ and $m(x,x)$ not defined
$X' = \{1\}$ with trivial multiplication and unit

X	1	x
1	1	x
x	x	DNE

X'	1
1	1

Then the map $f : X \to X', x \mapsto 1$ is a lax morphism but no strict morphism:

$(f \times f)m'$ **is defined everywhere,** mf **isn't**

Bags/Multisets

A **bag** (also known as a multiset) is a modification of the concept of a set that allows for repetitions, e.g.

$$\lbag 1, 2, 2, 2, 4 \rbag .$$

The union of bags adds the multiplicity of elements:

$$\lbag 1, 3, 3 \rbag \uplus \lbag 1, 2, 2, 3, 4 \rbag = \lbag 1, 1, 2, 2, 3, 3, 3, 4 \rbag$$

A bag a is a subbag of a bag b if the elments of a are in b and their multiplicites are at least as high in b as in a:

$$\lbag 1, 3, 3 \rbag \subset \lbag 1, 2, 3, 4, 3, 3 \rbag \not\subset \lbag 1, 2, 3, 4, 5 \rbag$$

Multiplying in a partial monoid

Given a partial commutative monoid X, a bag $w = \wr i_1, ..., i_n \int$ of numbers in $[k] := \{1, ..., k\}$, there is a map

$$\mu_w : X^k \to X \qquad (x_1, ..., x_k) \mapsto x_{i_1} \cdot ... \cdot x_{i_n} = \mu(x_{i_1}, \mu(x_{i_1}, ...\mu(x_{i_{n-1}}, x_{i_n})))$$

Especially

$$\mu_{\wr i \int} = \pi_i : X^k \to X,$$
$$\mu_{\emptyset} = u,$$

for $k = 2$

$$\mu_{\wr 1,2 \int} = \mu : X^2 \to X$$

and for $k = 1$

$$\mu_{\wr 1 \int} = 1_X : X \to X$$

These partial maps encode the monoid structure

In every partial monoid there are the maps

$$\bar{\mu}_{s_1} \dots \bar{\mu}_{s_l}(\mu_{b_1}, \dots, \mu_{b_n}) : X^k \to X^n$$

where $s_1, \dots, s_l, b_1, \dots, b_n$ are bags of numbers in $[k] = \{1, \dots, k\}$.

Definition (generic partial monoid)

Let the restriction category $\mathbb{B}$ consist of:

Objects: natural numbers 1,2,3...

Morphisms $k \to n$**:** pairs $(\mathcal{S}, \vec{w})$ where $\vec{w}$ is a n-tuple of bags in $[k]$ and $\mathcal{S}$ is a finite set of bags in $[k]$ fulfilling

- $\mathcal{S}$ is downwards closed under $\subset$
- $w_i \in \mathcal{S}$
- $\emptyset$ and singletons are in $\mathcal{S}$

Identity: $(\{\emptyset, \wr 1 \int, \dots \wr k \int\}, (\wr 1 \int, \dots \wr k \int))$

Composition: soon

Restriction: $\overline{(\mathcal{S}, \vec{w})} = (\mathcal{S}, (\wr 1 \int, \dots, \wr k \int)) : k \to k$

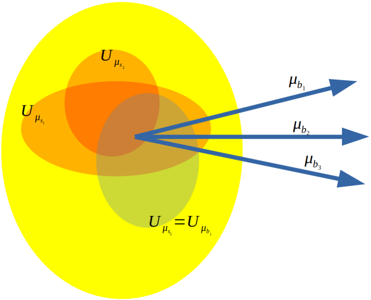
Restriction product: $(n) \times (m) = (n + m)$

Intuition

$$S = \{s_1, \ldots, s_l\} \longrightarrow \text{Where it is defined}$$

$$\bar{\mu}_{s_1} \ldots \bar{\mu}_{s_l}(\mu_{b_1}, \ldots, \mu_{b_n})$$

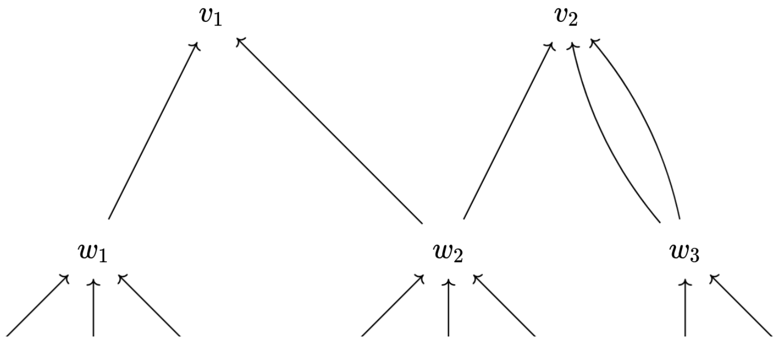
$$\vec{w} = (b_1, \ldots, b_n) \longrightarrow \text{What it does}$$



Composition

The composition $(\mathcal{S}, \vec{w})(\mathcal{R}, \vec{v})$ is supposed to resemble the compositions of

$$\bar{\mu}_{s_1}\dots\bar{\mu}_{s_l}(\mu_{b_1}, \dots, \mu_{b_n}) : X^k \to X^n$$



Composition (first component)

The composition $(\mathcal{S}, \vec{w})(\mathcal{R}, \vec{v})$ is supposed to resemble the compositions of

$$\bar{\mu}_{s_1}\ldots\bar{\mu}_{s_l}(\mu_{b_1}, \ldots, \mu_{b_n}) : X^k \to X^n$$

Given a n-tuple $\vec{w}$ of bags in m and a k-tuple $\vec{v}$ of bags in n, define

$$\vec{w}\vec{v} = (w_1, \ldots, w_n)(v_1, \ldots, v_k) = \left(\biguplus_{j \in v_1} w_j, \ldots, \biguplus_{j \in v_n} w_j \right)$$

which is a n-tuple of bags in m. Especially for $n = 1$, we get

$$\vec{w}r = \biguplus_{i \in r} w_i$$

Composition

The composition $(\mathcal{S}, \vec{w})(\mathcal{R}, \vec{v})$ is supposed to resemble the compositions of

$$\bar{\mu}_{s_1} \ldots \bar{\mu}_{s_l}(\mu_{b_1}, \ldots, \mu_{b_n}) : X^k \to X^n$$

For $f = (\mathcal{S}, \vec{w}) : [k] \to [n]$ and $g = (\mathcal{R}, \vec{v}) : [n] \to [m]$ the composition is

$$fg = (\downarrow (\mathcal{S} \cup \{\vec{w}r | r \in \mathcal{R}\}), \vec{w}\vec{v}).$$

Where $\downarrow$ is the downwards closure with respect to sub-bags and $\vec{w}r$ and $\vec{w}\vec{v}$ are the compositions of tuples.

Example

Define

$$\mu := (\mathcal{S}, \vec{w}) : 2 \to 1$$

with $\vec{w} = (\wr 1, 2 \int)$ and $\mathcal{S} = \{\emptyset, \wr 1 \int, \wr 2 \int, \wr 1, 2 \int\}$. Then

$$\begin{aligned}
&(\mu \times 1_1)\mu \\
=&(\{\emptyset, \wr 1 \int, \wr 2 \int, \wr 3 \int, \wr 1, 2 \int\}, (\wr 1, 2 \int, \wr 3 \int)); (\{\emptyset, \wr 1 \int, \wr 2 \int, \wr 1, 2 \int\}, (\wr 1, 2 \int)) \\
=&(\{\emptyset, \wr 1 \int, \wr 2 \int, \wr 3 \int, \wr 1, 2 \int, \wr 1, 2, 3 \int, \wr 2, 3 \int, \wr 1, 3 \int\}, (\wr 1, 2, 3 \int)) \\
=&\ldots \\
=&(1 \times \mu)\mu
\end{aligned}$$

The blue bags were added by the downwards closure.

Restriction functors

- A **restriction functor** is a functor F such that $\overline{F(f)} = F(\bar{f})$
- A **total strict natural transformation** ϕ between restriction functors $F, G : \mathbb{X} \to \mathbb{Y}$ is a family of total maps maps $(\phi_A : F(A) \to G(A))_{A \in \mathbb{X}_0}$ such that for every morphism $f : A \to B$ in $\mathbb{X}$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "= / \leq"] & F(B) \arrow[d, "\varphi_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

  commutes.
- A **total lax natural transformation** ϕ between restriction functors $F, G : \mathbb{X} \to \mathbb{Y}$ is a family of total maps maps $(\phi_A : F(A) \to G(A))_{A \in \mathbb{X}_0}$ such that for every morphism $f : A \to B$ in $\mathbb{X}$ the diagram commutes up to $\leq$.

Lawvere theory

A Lawvere theory for an algebraic structure is a category $\mathbb{A}$ such that for any category $\mathbb{X}$,

$$\mathrm{structure}(\mathbb{X}) \cong \mathit{Fun}_{\times}(\mathbb{A}, \mathbb{X})$$

Theorem

Suppose $\mathbb{X}$ is a restriction category with a restriction terminal object. Then

$$\mathrm{Fun}^{\mathrm{lax}}_{\times}(\mathbb{B}, \mathbb{X}) \cong \mathrm{LPCM}(\mathbb{X}) \qquad \textit{and} \qquad \mathrm{Fun}^{\mathrm{strict}}_{\times}(\mathbb{B}, \mathbb{X}) \cong \mathrm{SPCM}(\mathbb{X})$$

Future work

- **Noncommutativity:** Lists
- **Free constructions:**
  Let $\mathcal{U}$ be the forgetful functor from restriction categories to categories. Then it has a left-adjiont $\mathcal{F}$ that sends a category $\mathbb{X}$ to the restriction category with

  objects: $\mathrm{Ob}(\mathbb{X})$

  morphisms $A \to B$**:** pairs (f, M) of $f \in \mathbb{X}(A, B)$ and a finitely generated, right factor closed set M of morphisms with domain A containing 1_A
- **Operads?**

References

[1] J. R. B. Cockett and S. Lack, *Restriction categories I: categories of partial maps*, Theoretical Computer Science **270(1-2)** (2002), 223–259.

[2] J.R.B. Cockett and S. Lack, *Restriction categories II: Partial map classification*, Theo- retical Computer Science, **294** (2003), 61–102.

[3] J. R. B. Cockett and S. Lack, *Restriction categories iii: colimits, partial limits and extensivity*, Mathematical Structures in Computer Science **17** (2007), no. 4, 775–817.