# String diagrams for symmetric powers 

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## Introduction

This presentation is about symmetric powers in symmetric monoidal $\mathbb{Q}^{+}$-linear categories.

We provide a characterization of symmetric powers in terms of an algebraic structure that we call binomial graded bialgebras.

It provides some nice string diagrams.
We discuss propositions around this.
We finish by suggesting how to use these ideas for further work.

## Definition

A symmetric monoidal $\mathbb{Q}^{+}$-linear category is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ such that every hom-set $\mathcal{C}[A, B]$ is a $\mathbb{Q}^{+}$-module ${ }^{1}$, and moreover:
$--\otimes-: \mathcal{C}[A, B] \times \mathcal{C}[C, D] \rightarrow \mathcal{C}[A \otimes C, B \otimes D]$ is bilinear

- $-;-\mathcal{C}[A, B] \times \mathcal{C}[B, C] \rightarrow \mathcal{C}[A, C]$ is bilinear


## Example

$-\operatorname{Mod}_{\mathrm{R}}$ for $R$ any $\mathbb{Q}^{+}$-algebra (ie. a semiring ${ }^{2} R$ together with a semiring morphism $\mathbb{Q}^{+} \rightarrow R$ )

- Rel the category of sets and relations ${ }^{3}$
- $\mathrm{FVec}_{\mathrm{k}}$ the category of finite-dimensional vector spaces over a field $k$ of characteristic 0
- FRel the category of finite set and relations

In all the rest of the presentation $\mathcal{C}$ will be a symmetric monoidal $\mathbb{Q}^{+}$-linear category.

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\(1 \quad \mathbb{Q}^{+}=\)rational numbers \(\geq 0\)
2 semiring \(=\) ring without requiring negative numbers
3 In \(\operatorname{Rel}[A, B]\), + is the union of relations, 0 is the empty relation, \(\frac{n}{p} \cdot R:=R\) if \(\frac{n}{p} \neq 0\) and
\(0 . R:=0\). It gives a \(\mathbb{Q}^{+}\)-module because \(R+R=R\).
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For any object $A \in \mathcal{C}$ and every $n \geq 1$, we can form the $n^{\text {th }}$ tensor power $A^{\otimes n}$.
The $n^{\text {th }}$ symmetric power is a symmetrization of this tensor power. It can be interpreted in different ways which are all equivalents in this framework: ${ }^{4}$

$$
\begin{array}{r}
\text { An equalizer of } A^{\otimes n} \xrightarrow{\frac{\sigma}{\ldots}} A^{\otimes n} \\
\text { A coequalizer of } A^{\otimes n} \xrightarrow{\frac{\sigma}{\ldots}} A^{\otimes n} \\
\text { A splitting of the idempotent } \frac{1}{n!\sum_{\sigma \in \mathfrak{S}_{n}} \sigma: A^{\otimes n} \rightarrow A^{\otimes n}} \tag{3}
\end{array}
$$

Such a splitting is given by two maps $r_{n}: A^{\otimes n} \rightarrow A_{n}$ and $s_{n}: A_{n} \rightarrow A^{\otimes n}$ such that $r_{n} ; s_{n}=\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_{n}} \sigma$ and $s_{n} ; r_{n}=I d_{A_{n}}$.

## Proposition

Given objects $A$ and $A_{n} \in \mathcal{C}$, there are bijections between: an equalizer $s_{n}: A_{n} \rightarrow A^{\otimes n}$ a coequalizer $r_{n}: A^{\otimes n} \rightarrow A_{n}$, and a splitting $A^{\otimes n} \xrightarrow{r_{n}} A_{n} \xrightarrow{s_{n}} A^{\otimes n}$ of $\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_{n}} \sigma: A^{\otimes n} \rightarrow A^{\otimes n}$.

## Example

$-\operatorname{In} \operatorname{Mod}_{R}\left(R\right.$ a $\mathbb{Q}^{+}$-algebra $)$, the $n^{\text {th }}$ symmetric power can be seen equivalently as the subspace $\left(A^{\otimes n}\right)^{\mathfrak{G}_{n}}$ of vectors invariants by permutation or as the quotient $\left(A^{\otimes n}\right)_{\mathfrak{S}_{n}}$ of $A^{\otimes n}$ by the $n$ ! permutations.

- In Rel, the $n^{\text {th }}$ symmetric power of $A$ is the set $\mathcal{M}_{n}(A)$ of multisets of $n$ elements in $A . r_{n}$ sends any tuple ( $a_{1}, \ldots, a_{n}$ ) to the multiset $\left[a_{1}, \ldots, a_{n}\right], s_{n}$ relates any multiset $\left[a_{1}, \ldots, a_{n}\right]$ to all the tuples $\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right)$ for every $\sigma \in \mathfrak{S}_{n} . r_{n} ; s_{n}$ relates any tuple $\left(a_{1}, \ldots, a_{n}\right)$ to all the tuples $\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right)$ for every $\sigma \in \mathfrak{S}_{n}$.

It provides our first definition of a family $\left(A_{n}\right)_{n \geq 1}$ of symmetric powers:

## Definition

In a symmetric monoidal $\mathbb{Q}^{+}$-linear category, a permutation splitting is given by:

- a family $\left(A_{n}\right)_{n \geq 1}$ of objects
- a family $\left(r_{n}: A_{1}^{\otimes n} \rightarrow A_{n}\right)_{n \geq 2}$ of morphisms
- a family $\left(s_{n}: A_{n} \rightarrow A_{1}^{\otimes n}\right)_{n \geq 2}$ of morphisms
such that:
$-r_{n} ; s_{n}=\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_{n}} \sigma: A^{\otimes n} \rightarrow A^{\otimes n}$
$-s_{n} ; r_{n}=I d_{A_{n}}$


## Definition

A morphism of permutation splittings $\left(A_{n}\right)_{n \geq 1} \rightarrow\left(B_{n}\right)_{n \geq 1}$ is given by a family $\left(f_{n}: A_{n} \rightarrow B_{n}\right)_{n \geq 1}$ such that any of these three equivalent conditions are verified:

$$
\begin{align*}
& A_{1}^{\otimes n} \xrightarrow{f_{1}^{\otimes n}} B_{1}^{\otimes n} \\
& r_{n}^{A} \downarrow  \tag{4}\\
& \downarrow{ }^{r_{n}^{B}} \\
& A_{n} \xrightarrow[f_{n}]{ } B_{n} \\
& f_{n}=r_{n}^{A} ; f_{1}^{\otimes n} ; s_{n}^{B}  \tag{5}\\
& A_{n} \xrightarrow{s_{n}^{A}} A_{1}^{\otimes n} \\
& f_{n} \downarrow \quad \downarrow_{1}^{\otimes n}  \tag{6}\\
& B_{n} \xrightarrow[s_{n}^{B}]{ } B_{1}^{\otimes n}
\end{align*}
$$

## Proposition

The category of permutation splittings is isomorphic to the category of permutations splittings $\left(\left(A_{n}\right)_{n \geq 1},\left(r_{n}\right)_{n \geq 1},\left(s_{n}\right)_{n \geq 1}\right)$ and morphisms $f_{1}: A_{1} \rightarrow B_{1}$ ( $=$ the reduced category of permutation splittings).

Given a permutation splitting $\left(\left(A_{n}\right)_{n \geq 1},\left(r_{n}\right)_{n \geq 1},\left(s_{n}\right)_{n \geq 1}\right)$, we can define $\left(\nabla_{n, p}: A_{n} \otimes A_{p} \rightarrow A_{n+p}\right)_{n, p \geq 1},\left(\Delta_{n, p}: A_{n+p} \rightarrow A_{n} \otimes A_{p}\right)_{n, p \geq 1}$ by:


These equations are then verified: ${ }^{5} 6$


5 first one for every $n, p, q, r \geq 1$ such that $n+p=q+r$, second one for every $n, p \geq 1$ where we note:


It provides our second definition of a family $\left(A_{n}\right)_{n \geq 1}$ of symmetric powers:

## Definition

In a symmetric monoidal $\mathbb{Q}^{+}$-linear category, a binomial graded bialgebra is given by:

- a family $\left(A_{n}\right)_{n \geq 1}$ of objects
- a family $\left(\nabla_{n, p}: A_{n} \otimes A_{p} \rightarrow A_{n+p}\right)_{n, p \geq 1}$ of morphisms
- a family $\left(\Delta_{n, p}: A_{n+p} \rightarrow A_{n} \otimes A_{p}\right)_{n, p \geq 1}$ of morphisms such that:



## Proposition

Every binomial graded bialgebra is biassociative and bicommutative.

Given a binomial graded bialgebra, we can define:


## Proposition

Given a family $\left(A_{n}\right)_{n \geq 1}$ of objects, the constructions between the families $\left(r_{n}\right)_{n \geq 1},\left(s_{n}\right)_{n \geq 1}$ which define a permutation splitting and the families $\left(\nabla_{n, p}: A_{n} \otimes A_{p} \rightarrow A_{n+p}\right)_{n, p \geq 1},\left(\Delta_{n, p}: A_{n+p} \rightarrow A_{n} \otimes A_{p}\right)_{n, p \geq 1}$ which define a binomial graded bialgebra provide a bijection between splitting idempotents with underlying objects $\left(A_{n}\right)_{n \geq 1}$ and binomial graded bialgebras with underlying objects $\left(A_{n}\right)_{n \geq 1}$.

## Corollary

In a symmetric monoidal $\mathbb{Q}^{+}$-linear category, a family $\left(A_{n}\right)_{n \geq 1}$ verifies that $A_{n}$ is the $n^{\text {th }}$ symmetric power of $A_{1}$ iff it can be equipped with a structure of binomial graded bialgebra.

How to prove that in a binomial graded bialgebra, $A_{n}$ is the $n^{\text {th }}$ symmetric power of $A_{1}$ ?

We must show that if we define

then

ie. we must prove that:


The second equality is not very difficult to prove. One of our axiom is that


It then follows by using this for every $1 \leq k \leq n-1$.
The first one is a bit more difficult. Define


We can then prove by induction on $(q, r) \in \mathbb{N}_{\geq 2}^{\times 2}$ that

for positive integers such that $n_{1}+\ldots+n_{q}=p_{1}+\ldots+p_{r}$.
The sum is indexed by (0-1)-matrices with constrained sums of lines and sums of columns.

We then put $q=r:=n$, all inputs and outputs equal to 1 and deduce that


The sum is indexed by $(0,1)$-matrices such that there is exacly one 1 in every column and one 1 in every line.

This is what we call a permutation matrix!

We have proved that

we then multiply each side by $\frac{1}{n!}$ to obtain the desired equality.

## Morphisms of binomial graded bialgebras

## Proposition

If $\left(f_{n}: A_{n} \rightarrow B_{n}\right)_{n \geq 1}$ is a family of morphisms between two binomial graded biagebras, these conditions are equivalent: ${ }^{7}$

$$
\begin{align*}
& A_{n} \otimes A_{p} \xrightarrow{\nabla_{n, p}^{A}} A_{n+p} \\
& f_{n} \otimes f_{p} \downarrow \downarrow^{f_{n+p}}  \tag{7}\\
& B_{n} \otimes B_{p} \xrightarrow[\nabla_{n, p}^{B}]{ } B_{n+p} \\
& f_{n}=r_{n}^{A} ; f_{1}^{\otimes n} ; s_{n}^{B}  \tag{8}\\
& A_{n+p} \xrightarrow{\Delta_{n, p}^{A}} A_{n} \otimes A_{p} \\
& f_{n+p} \downarrow \quad \downarrow^{f_{n} \otimes f_{p}}  \tag{9}\\
& B_{n+p} \xrightarrow[\Delta_{n, p}^{B}]{ } B_{n} \otimes B_{p}
\end{align*}
$$

If any of these conditions is verified, we say that $\left(f_{n}\right)_{n \geq 1}$ is a morphism of binomial graded bialgebras.

## Corollary

If $R$ is a $\mathbb{Q}^{+}$-module, if $f: R\left[Y_{1}, \ldots, Y_{n}\right] \rightarrow R\left[X_{1}, \ldots, X_{p}\right]$ is a linear map which preserves the degree of homogeneous polynomials, then it is a morphism of bialgebras iff it is a morphism of algebras iff it is a morphism of coalgebras and there is exactly one such morphism for every linear map $R . Y_{1} \oplus \ldots \oplus R . Y_{n} \rightarrow R . X_{1} \oplus \ldots \oplus R . X_{p}$.

## Example

For every $\sigma \in \mathfrak{S}_{n}$, the linear map $f_{\sigma}: R\left[X_{1}, \ldots, X_{n}\right] \rightarrow R\left[X_{1}, \ldots, X_{n}\right]$ which sends $X_{1} \mapsto X_{\sigma(1)}, \ldots, X_{n} \mapsto X_{\sigma(n)}$ is such a morphism. A polynomial $P \in R\left[X_{1}, \ldots, X_{n}\right]$ is called symmetric iff it is invariant by $f_{\sigma}$ for every $\sigma \in \mathfrak{S}_{n}$.

## Proposition

The category of binomial graded bialgebras is isomorphic to the category of permutation splittings, to the reduced category of binomial graded bialgebras and to the reduced category of binomial graded bialgebras.

- Between the category of binomial graded bialgebras and of permutation splittings, the isomorphism sends $\left(f_{n}\right)_{n \geq 1}$ to $\left(f_{n}\right)_{n \geq 1}$
- from a non-reduced category to a reduced one, it sends $\left(f_{n}\right)_{n \geq 1}$ to $f_{1}$
- from a reduced category to a non-reduced one, it sends $f_{1}$ to $\left(f_{n}\right)_{n \geq 1}$ where $f_{n}: A_{n} \rightarrow B_{n}$ is defined by $f_{n}=r_{n}^{A} ; f_{1}^{\otimes n} ; s_{n}^{B}$.


## References

If you want to understand how it is related to differentiation and to logic, check the section "Symmetric powers" of our paper:

Graded Differential Categories and Graded Differential Linear Logic (JS and V.)

If you want to read this story, check in a few weeks the preprint:
String diagrams for symmetric powers
(V.)

## Future work

There is a lot to do by changing the type of powers and/or the enrichment:

- Binomial graded bialgebras in symmetric monoidal $\mathbb{N}$-linear categories: The exact same definition can be written in this context. There then seems to be several different kinds of binomial graded bialgebras such as symmetric algebras, divided power algebras and exterior algebras in $V e C_{\mathbb{Z}_{2}}$.
- Exterior powers in symmetric monoidal $\mathbb{Q}$-linear categories: Define binomial graded-commutative bialgebras and prove that they characterize exterior powers in this context.
- Schur functors in symmetric monoidal $\mathbb{Q}$-linear categories: In any symmetric monoidal $\mathbb{Q}$-linear category, for every $n \geq 1$ and partition $\lambda \vdash n$, there is an idempotent natural transformation $s_{\lambda}:{ }_{-}{ }^{\otimes n} \rightarrow_{-}{ }^{\otimes n}$. The Schur functor $S^{\lambda}: \mathcal{C} \rightarrow \mathcal{C}$ is defined as an intermediate object in a splitting of this idempotent ${ }^{8}$. So we then have two natural transformations $r_{\lambda}:{ }_{-}{ }^{\otimes n} \rightarrow S^{\lambda}$ and $s_{\lambda}: S^{\lambda} \rightarrow{ }_{-}^{\otimes n}$. What are the equations verified by combining the natural transformations $\nabla_{\lambda, \mu}: S^{\lambda} \otimes S^{\mu} \rightarrow S^{\lambda+\mu}$, $\Delta_{\lambda, \mu}: S^{\lambda+\mu} \rightarrow S^{\lambda} \otimes S^{\mu} ?^{9}$

[^0]Thank you for your attention!


[^0]:    8 The partition $n \vdash n$ gives the $n^{\text {th }}$ symmetric power and the partition $1, \ldots, 1 \vdash n$ gives the $n^{\text {th }}$ exterior power!

