Ind Tangent Categories

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The Punchline Comes First

Theorem

Let $(\mathscr{C}, \mathbb{T})$ be a tangent category. Then there is a tangent structure $\mathsf{Ind}(\mathbb{T})$ on the ind-category $\mathsf{Ind}(\mathscr{C})$ induced by $\mathbb{T}$ such that $(\mathsf{Ind}(\mathscr{C}), \mathsf{Ind}(\mathbb{T}))$ is a tangent category.

The Punchline Comes First

Theorem

Let $(\mathscr{C}, \mathbb{T})$ be a tangent category. Then there is a tangent structure $\mathrm{Ind}(\mathbb{T})$ on the ind-category $\mathrm{Ind}(\mathscr{C})$ induced by $\mathbb{T}$ such that $(\mathrm{Ind}(\mathscr{C}), \mathrm{Ind}(\mathbb{T}))$ is a tangent category.
Furthermore, if $(F, \alpha) : (\mathscr{C}, \mathbb{T}) \to (\mathscr{D}, \mathbb{S})$ is a tangent morphism of tangent categories then $(\mathrm{Ind}(F), \hat{\alpha}) : (\mathrm{Ind}(\mathscr{C}), \mathrm{Ind}(\mathbb{T})) \to (\mathrm{Ind}(\mathscr{D}), \mathrm{Ind}(\mathbb{S}))$ is a tangent morphism as well. Finally, α is strong if and only if $\hat{\alpha}$ is strong.

Why Ind-Cats?

For Real (and p-adic): Why Though?

My intuition lies in schemey things. There is an equivalence of categories between the category of formal schemes and the ind-category of schemes. A formal scheme is on one hand a formal filtered cocompletion of a schemes and on the other hand a completion of a scheme along a closed subscheme. We have tangents for schemes by work of Geoff and JS in [3]. Now we should be able to formally complete these tangents along closed subschemes to get tangents of formal functions.

What is an Ind-Category

In one line: if $\mathscr{C}$ is a category then $\mathrm{Ind}(\mathscr{C})$ is the free cocompletion of $\mathscr{C}$. It was first discovered and studied by Grothendieck and Verdier in [1, Exposé I.8.2-9].

Our Friend, the Ind-Category

The Presheaf Representation

Let $\mathscr{C}$ be a category and let $\mathbf{y} : \mathscr{C} \to [\mathscr{C}^{\mathsf{op}}, \underline{\mathsf{Set}}]$ be its Yoneda embedding. The Density Lemma says that for any presheaf $P \in [\mathscr{C}^{\mathsf{op}}, \underline{\mathsf{Set}}]$ there is an isomorphism

$$P \cong \int^{X \in \mathscr{C}} P(X) \times \mathscr{C}(-, X) \cong \operatorname*{colim}_{(\mathbf{y}(X) \to P) \in (\mathbf{y} \downarrow P)} \mathscr{C}(-, X)$$

where $\mathbf{y} \downarrow P$ denotes the comma category of $\mathbf{y}$ over P.
This realizes $[\mathscr{C}^{\mathsf{op}}, \underline{\mathsf{Set}}]$ as the free cocompletion of $\mathscr{C}$ as it says every presheaf is a colimit of representable functors.

Our Friend, the Ind-Category

The Presheaf Representation

To pick out the free filtered cocompletion, we only take those presheaves which come from filtered colimits!

Our Friend, the Ind-Category

The Presheaf Representation

Define the presheaf Ind-cocompletion of $\mathscr{C}$ as follows:

- Objects: Presheaves $P \in [\mathscr{C}^{\mathrm{op}}, \underline{\mathrm{Set}}]_0$ which have an isomorphism

$$P \cong \operatorname*{colim}_{i \in I} \mathscr{C}(-, X_i)$$

  with I a filtered category.
- Morphisms, Identities, and Composition: As in $[\mathscr{C}^{\mathrm{op}}, \underline{\mathrm{Set}}]$.

Our Friend, the Ind-Category

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We write this category as $\mathrm{Ind}_{\mathrm{PSh}}(\mathscr{C})$.

Our Friend, the Ind-Category

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- Morphisms, Identities, and Composition: As in $[\mathscr{C}^{op}, \underline{\mathrm{Set}}]$.

An important question: Is this invariant under equivalence? How can we extract structure?

Our Friend, the Ind-Category

The Representation Agnostic Construction

Let $\mathscr{C}$ be a category. An ind-object of $\mathscr{C}$ is a functor

$$F : I \to \mathscr{C}$$

where I is a filtered category. These make up the objects of $\mathrm{Ind}(\mathscr{C})$.

Our Friend, the Ind-Category

The Representation Agnostic Construction

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Notation

We will often write ind-objects in $\mathscr{C}$ as tuples $\underline{X} = (X_i)_{i \in I}$ where $X_i := F(i)$ for all $i \in I_0$. This leaves both the transition morphisms and functor F implicit, so we'll only use this when it won't cause confusion. We'll also write

$$(X_i)_{i \in I} \leftrightsquigarrow F : I \to \mathscr{C}$$

to denote moving between the two representations of objects.

Our Friend, the Ind-Category

The Representation Agnostic Construction

For ind-objects $(X_i), (Y_j)$ a morphism $\rho : (X_i) \to (Y_j)$ is somewhat trickier to define. We will not go through this too explicitly here, but we describe what our hom-set is:

$$\mathrm{Ind}(\mathscr{C})\left((X_i)_{i \in I}, (Y_j)_{j \in J}\right) := \lim_{i \in I} \left(\operatorname*{colim}_{j \in J} \mathscr{C}\left(X_i, Y_j\right) \right)$$

Our Friend, the Ind-Category

The Representation Agnostic Construction

The ind-category $\mathrm{Ind}(\mathscr{C})$ is defined by:

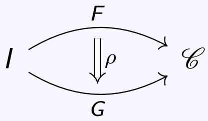
- Objects: Ind-objects $\underline{X} = (X_i)$ for filtered categories and functors $F : I \to \mathscr{C}$.
- Morphisms, Composition, Identities: Induced by the assignment

$$\mathrm{Ind}(\mathscr{C})\big((X_i)_{i\in I}, (Y_j)_{j\in J}\big) := \lim_{i\in I}\left(\operatorname*{colim}_{j\in J} \mathscr{C}(X_i, Y_j)\right).$$

Our Friend, the Ind-Category

The Representation Agnostic Construction

When $(X_i) \leftrightsquigarrow F : I \to \mathscr{C}$ and $(Y_j) \leftrightsquigarrow G : J \to \mathscr{C}$ are ind-objects with the indexing category $I = J$ we can describe the morphisms $(X_i) \to (Y_j)$ as natural transformations ρ:



In this case the hom's in $\mathrm{Ind}(\mathscr{C})$ are simply the colimit of the hom's $\rho_i : X_i \to Y_i$ induced by ρ along the diagrams. We write these morphisms as

$$(\rho_i) : (X_i) \to (Y_i), \qquad \rho_i : X_i \to Y_i \leftrightsquigarrow \rho_i : F(i) \to G(i)$$

Some Ind-Facts

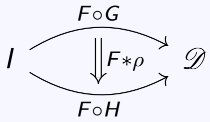
(Pseudo)Functoriality

The $\mathrm{Ind}(-)$ construction can be extended to functors in the following sense: If there is a functor $F : \mathscr{C} \to \mathscr{D}$ then there is a functor $\mathrm{Ind}(F) : \mathrm{Ind}(\mathscr{C}) \to \mathrm{Ind}(\mathscr{D})$. It is defined by the assignments below:

Some Ind-Facts

(Pseudo)Functoriality

- On objects: if $(X_i) \leftrightsquigarrow G : I \to \mathscr{C}$ then $\mathrm{Ind}(F)(X_i) := (FX_i) \leftrightsquigarrow F \circ G : I \to \mathscr{D}$.
- On specific morphisms: if $G, H : I \to \mathscr{C}$ are functors for I filtered and $\rho : F \Rightarrow G$ then $\mathrm{Ind}(F)(\rho) := F * \rho$ as in the two-cell:



Some Ind-Facts

(Pseudo)Functoriality

- On generic morphisms: the assignment on morphisms is induced by the following natural map. For any functors $G : I \to \mathscr{C}$ and $H : J \to \mathscr{C}$ for I, J filtered there is a natural map

$$\theta_F^{i,j} : \mathscr{C}(Gi, Hj) \to \mathscr{D}\left(F(Gi), F(Hj)\right).$$

The assignment of $\mathrm{Ind}(F)$ is induced by taking the limit of the filtered colimit of the $\theta_F^{i,j}$, i.e., by

$$\lim_{i \in I}\left(\operatorname*{colim}_{j \in J} \mathscr{C}(Gi, Hj)\right) \xrightarrow{\theta_F^{I,J}} \lim_{i \in I}\left(\operatorname*{colim}_{j \in J} \mathscr{D}\left(F(Gi), F(Hj)\right)\right).$$

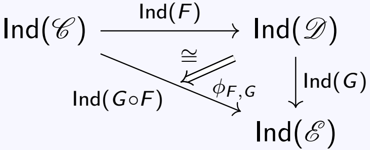
Some Ind-Facts

(Pseudo)Functoriality

For any composable functors

$$\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E}$$

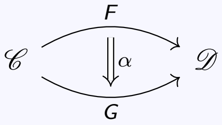
there is a compositor natural isomorphism $\phi_{F,G}$ which we visualize in the invertible 2-cell



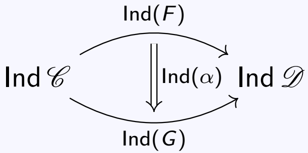
Some Ind-Facts

(Pseudo)Functoriality

For any natural transformation



there is a corresponding natural transformation of ind-functors:



Some Ind-Facts

(Pseudo)Functoriality

To define the ind-transformation $\mathrm{Ind}(\alpha) : \mathrm{Ind}(F) \Rightarrow \mathrm{Ind}(G)$ note that we need maps $\mathrm{Ind}(F)(X_i) \to \mathrm{Ind}(G)(X_i)$ for any $(X_i) \in \mathrm{Ind}(\mathscr{C})_0$. But if $(X_i) \leftrightsquigarrow H : I \to \mathscr{C}$ then the horizontal composite

$$I \underset{G\circ H}{\overset{F\circ H}{\underset{\Downarrow \alpha * H}{\rightrightarrows}}} \mathscr{D}$$

determines $\mathrm{Ind}(\alpha)$ as the natrual transformation with components $\mathrm{Ind}(\alpha)_{X_i} = (\alpha_{X_i}) : (FX_i) \to (GX_i)$.

Some Ind-Facts

(Pseudo)Functoriality

Putting it together we get an Ind-pseudofunctor

$$\mathrm{Ind} : \mathfrak{Cat} \to \mathfrak{Cat}$$

where $\mathfrak{Cat}$ is the (strict) 2-category of categories.
Also there is an equivalence of categories $L : \mathrm{Ind}(\mathscr{C}) \xrightarrow{\simeq} \mathrm{Ind}_{\mathrm{PSh}}(\mathscr{C})$ given by

$$L\big((X_i)_{i\in I}\big) := \operatorname*{colim}_{i\in I} \mathbf{y}(X_i).$$

The Tangent Category

Tangent Categories and Tangent Morphisms

Tangent categories are a categorification of the tangent bundle functor $T : \mathbf{SMan} \to \mathbf{SMan}$ of smooth (real) manifolds and its properties. Rediscovered and phrased in a modern language by Robin and Geoff in [2], these give categorical tools with which to use differential geometric techniques and reasoning in myriad and wide-reaching settings.

Tangent Categories and Tangent Morphisms

Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ has a tangent structure $\mathbb{T} = (T, p, \text{add}, 0, \ell, c)$ when:

Tangent Categories and Tangent Morphisms

- $T : \mathscr{C} \to \mathscr{C}$ is an endofunctor, $p : T \Rightarrow \mathrm{id}_{\mathscr{C}}$ is a natural transformation, the pullback powers

$$\begin{array}{ccc} T_2X & \longrightarrow & TX \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p_X} \\ TX & \xrightarrow[p_X]{} & X \end{array}$$

  exist and each functor T^n preserves these pullback powers (for any $n \in \mathbb{N}$).

Tangent Categories and Tangent Morphisms

- add : $T_2 \Rightarrow T$ and $0 : \mathrm{id}_{\mathscr{C}} \Rightarrow T$ are natural transformations such that each map $p_X : TX \to X$ is a commutative monoid internal to $\mathscr{C}_{/X}$ with addition and unit given by add_X and 0_X, respectively.

The Tangent Category

Tangent Categories and Tangent Morphisms

- $\ell : T \Rightarrow T^2$ is a natural transformation (called the vertical lift) which make $(\ell_X, 0_X)$ into a morphism of additive bundles in $\mathscr{C}$ for all objects X.
- $c : T^2 \Rightarrow T^2$ is a natural transformation (called the canonical flip) which makes (c_X, id_{TX}) into a bundle map in $\mathscr{C}$ for any object X.

The Tangent Category

Tangent Categories and Tangent Morphisms

- The equations $c^2 = \mathrm{id}_{T^2}$ and $c \circ \ell = \ell$ hold. Additionally, the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\ell} & T^2 \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle T*\ell} \\ T^2 & \xrightarrow[\ell*T]{} & T^3 \end{array} \qquad \begin{array}{ccccc} T^3 & \xrightarrow{T*c} & T^3 & \xrightarrow{c*T} & T^3 \\ {\scriptstyle c*T}\downarrow & & & & \downarrow{\scriptstyle c*T} \\ T^3 & \xrightarrow[T*c]{} & T^3 & \xrightarrow[c*T]{} & T^3 \end{array}$$

$$\begin{array}{ccccc} T^2 & \xrightarrow{\ell*T} & T^3 & \xrightarrow{T*c} & T^3 \\ {\scriptstyle c}\downarrow & & & & \downarrow{\scriptstyle c*T} \\ T^2 & & \xrightarrow[T*\ell]{} & & T^3 \end{array}$$

commute in $[\mathscr{C}, \mathscr{C}]$.

The Tangent Category

Tangent Categories and Tangent Morphisms

- For any $X \in \mathscr{C}_0$ the diagram

$$T_2X \xrightarrow{(T*\text{add})_X \circ \langle \ell \circ \pi_1, 0_{TX} \circ \pi_2 \rangle} T^2X \underset{0_X \circ p_X \circ p_{TX}}{\overset{T(p_X)}{\rightrightarrows}} TX$$

  is an equalizer diagram.

Tangent Categories and Tangent Morphisms

If $\mathscr{C}$ is a category with a tangent structure $\mathbb{T}$, we say that the pair $(\mathscr{C}, \mathbb{T})$ is a tangent category. By abuse of notation we will sometimes say that $\mathscr{C}$ is a tangent category and leave the tangent structure implicit until needed.

The Tangent Category

Tangent Categories and Tangent Morphisms

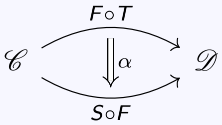
Let $(\mathscr{C}, \mathbb{T})$ and $(\mathscr{D}, \mathbb{S})$ be tangent categories with

$$\mathbb{T} = (T, p, \mathrm{add}_{\mathbb{T}}, 0_{\mathbb{T}}, \ell_{\mathbb{T}}, c_{\mathbb{T}})$$

and

$$\mathbb{S} = (S, q, \mathrm{add}_{\mathbb{S}}, 0_{\mathbb{S}}, \ell_{\mathbb{S}}, c_{\mathbb{S}}).$$

We say that a pair $(F, \alpha) : (\mathscr{C}, \mathbb{T}) \to (\mathscr{D}, \mathbb{S})$ is a tangent morphism if $F : \mathscr{C} \to \mathscr{D}$ is a functor, α is a natural transformation



such that the diagrams:

The Tangent Category

Tangent Categories and Tangent Morphisms

$$\begin{array}{ccc} F \circ T & \xrightarrow{\alpha} & S \circ F \\ & \searrow^{F*p} & \downarrow{\scriptstyle q*F} \\ & & F \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{F*0_{\mathbb{T}}} & F \circ T \\ & \searrow^{0_{\mathbb{S}}*F} & \downarrow{\scriptstyle \alpha} \\ & & S \circ F \end{array} \qquad \begin{array}{ccc} F \circ T_2 & \xrightarrow{F*\alpha_2} & S_2 \circ F \\ {\scriptstyle F*\mathsf{add}_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle \mathsf{add}_{\mathbb{S}}*F} \\ F \circ T & \xrightarrow[\alpha]{} & S \circ F \end{array}$$

and

$$\begin{array}{ccc} F \circ T & \xrightarrow{\alpha} & S \circ F \\ {\scriptstyle F*\ell_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle \ell_{\mathbb{S}}*F} \\ F \circ T^2 & \xrightarrow[(S*\alpha)\circ(\alpha*T)]{} & S^2 \circ F \end{array} \qquad \begin{array}{ccc} F \circ T^2 & \xrightarrow{(S*\alpha)\circ(\alpha*T)} & S^2 \circ F \\ {\scriptstyle F*c_{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle c_{\mathbb{S}}*F} \\ F \circ T^2 & \xrightarrow[(S*\alpha)\circ(\alpha*T)]{} & S^2 \circ F \end{array}$$

commute. If α is a natural isomorphism then we say that (F, α) is a strong tangent morphism.

The Tangent Category

Tangent Categories and Tangent Morphisms

We write **Tan** as the 1-category of tangent categories with tangent morphisms and $\mathfrak{Tan}$ as the 2-category of tangent categories, tangent morphisms, and natural transformations.

Defining the Ind-Tangent Structure

The Ind-Tangent Structure

Fix a tangent category $(\mathscr{C}, \mathbb{T})$ and tangent structure $\mathbb{T} = (T, p, \mathrm{add}, 0, \ell, c)$. Then we define $\mathrm{Ind}(\mathbb{T})$ as:

$$\mathrm{Ind}(\mathbb{T}) = (\mathrm{Ind}(T), \mathrm{Ind}(p), \widehat{\mathrm{add}}, \hat{0}, \hat{\ell}, \hat{c}).$$

The functor $\mathrm{Ind}(T)$, the bundle map $\mathrm{Ind}(p)$, and the unit $\hat{0}$ are simply the Ind-pseudofunctor applied to T, p, and 0, respectively. But what about the rest?

Defining the Ind-Tangent Structure

The Ind-Addition

We can prove, using the equivalence of categories $\mathrm{Ind}(\mathscr{C}) \simeq \mathrm{Ind}_{\mathrm{PSh}}(\mathscr{C})$, that there is an isomorphism of functors $\mathrm{Ind}(T)_2 \cong \mathrm{Ind}(T_2)$. The transformations $\widehat{\mathrm{add}}$ is then given by:

$$\begin{array}{ccc} \mathrm{Ind}(T)_2 & \xrightarrow{\cong} & \mathrm{Ind}(T_2) \\ & \underset{\widehat{\mathrm{add}}}{\searrow} & \downarrow{\scriptstyle \mathrm{Ind}\,\mathrm{add}} \\ & & \mathrm{Ind}(T) \end{array}$$

Defining the Ind-Tangent Structure

The Ind-Lift

To describe the ind-lift $\hat{\ell}$ we first note that because $\operatorname{Ind}(-)$ is a pseudofunctor the compositor gives a natural isomorphism

$$\phi_{T,T} : \operatorname{Ind}(T)^2 \xRightarrow{\cong} \operatorname{Ind}(T^2).$$

We define $\hat{\ell}$ as the composite:

$$\begin{array}{ccc} \operatorname{Ind}(T)^2 & \xrightarrow{\phi_{T,T}} & \operatorname{Ind}(T^2) \\ & \searrow^{\hat{\ell}} & \downarrow{\scriptstyle \operatorname{Ind}(\ell)} \\ & & \operatorname{Ind}(T) \end{array}$$

Defining the Ind-Tangent Structure

The Ind-Flip

To describe $\hat{c} : \mathrm{Ind}(T)^2 \Rightarrow \mathrm{Ind}(T)^2$ we again need to use the compositor $\phi_{T,T}$. It is defined via the diagram:

$$\begin{array}{ccc} \mathrm{Ind}(T)^2 & \xrightarrow{\phi_{T,T}} & \mathrm{Ind}(T^2) \\ \hat{c}\downarrow & & \downarrow \mathrm{Ind}(c) \\ \mathrm{Ind}(T)^2 & \xleftarrow[\phi_{T,T}^{-1}]{} & \mathrm{Ind}(T^2) \end{array}$$

The Functoriality

Ind-Morphisms

Let $(\mathscr{C}, \mathbb{T})$ and $(\mathscr{D}, \mathbb{S})$ be tangent categories with a tangent morphism $(F, \alpha) : (\mathscr{C}, \mathbb{T}) \to (\mathscr{D}, \mathbb{S})$. Then there is a tangent morphism

$$(\mathrm{Ind}(F), \hat{\alpha}) : (\mathrm{Ind}(\mathscr{C}), \mathrm{Ind}(\mathbb{T})) \to (\mathrm{Ind}(\mathscr{D}), \mathrm{Ind}(\mathbb{S}))$$

which is strong if and only if (F, α) is strong.

Defining $\hat{\alpha}$

The natural transformation $\hat{\alpha} : \mathrm{Ind}(F) \circ \mathrm{Ind}(T) \Rightarrow \mathrm{Ind}(S) \circ \mathrm{Ind}(F)$ is defined via the diagram:

$$\begin{array}{ccc} \mathrm{Ind}(F) \circ \mathrm{Ind}(T) & \xrightarrow{\phi_{T,F}} & \mathrm{Ind}(F \circ T) \\ {\scriptstyle \hat{\alpha}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ind}(\alpha)} \\ \mathrm{Ind}(S) \circ \mathrm{Ind}(F) & \xleftarrow[\phi_{F,S}^{-1}]{} & \mathrm{Ind}(S \circ F) \end{array}$$

Sketch of the strength.

First if $\hat{\alpha}$ is strong, it is strong on every constant object $(X) \leftrightsquigarrow X : \mathbb{1} \to \mathscr{C}$ and so $\hat{\alpha} = \phi_{F,S}^{-1} \circ \alpha_X \circ \phi_{T,F}$ is an isomorphism. Since the ϕ are isomorphisms, we get that α_X is an isomorphism for any X.
If on the other hand α is a natural isomorphism, then $\mathsf{Ind}(\alpha)$ is an isomorphism as well. It is then immediate that $\hat{\alpha} = \phi_{F,S}^{-1} \circ \mathsf{Ind}(\alpha) \circ \phi_{T,F}$ is an isomorphism. □

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The Last Slide

The End

Thanks for coming and listening everybody!