(Draft) Notes for Algebraic Geometry: A different path up the mountain

Geoff Cruttwell

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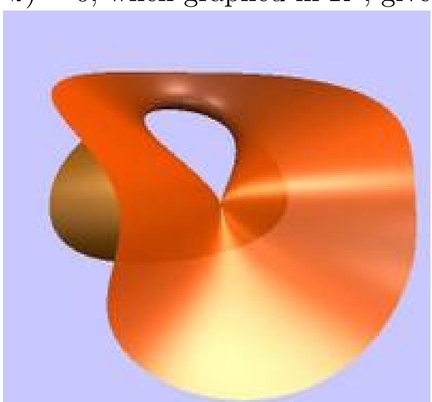
Abstract

These notes are intended as an alternative introduction to some of the ideas in modern algebraic geometry. The main requirements are some basic knowledge of category theory (opposite category, monomorphisms/epimorphisms/isomorphisms, terminal objects, pullbacks) and basic knowledge of ring theory (ideals, quotients, fields).

Contents

1 Introduction

Algebraic geometry, the study of the geometry of solutions of polynomial equations, is a beautiful subject. It is fascinating how relatively simple equations like $xz + y^2(x + y + z) = 0$, when graphed in $\mathbb{R}^3$, give incredible images like



And there is an amazing interplay, which we will see, between the geometry of these objects and algebraic concepts like ideals, localizations, and formal power series. As we go through these notes, we will build up the following table of equivalences between geometric concepts on the one side and algebraic ones on the other:

Geometric object	Algebraic object
Affine n-space	Polynomial ring $R[x_1, \ldots, x_n]$
Variety over R	Quotient ring $R[x_1, \ldots, x_n]/I$
Solution of X in R	Ring morphism $X \longrightarrow R$
Open subset in X not including f	Localization of X at f
Germ at variety represented by P	Localization of X at prime ideal P
Generic point of variety represented P	Residue field $k(P)$ of prime ideal P
Analytic neighborhood at origin	Formal power series ring $R[[x_1, \ldots x_n]]$
Pullback (eg., intersection, fibre...)	Tensor product $B \otimes_A C$
Tangent bundle	Symmetric algebra of Kahler differentials

But for all its beauty and fascinating ideas, modern algebraic geometry is intimidating, to say the least. Open up almost any modern algebraic geometry book and one is immediately hit by a wall of definitions which can feel very unmotivated (at least, to me): prime ideals as points, the Zariski topology, sheaves, locally ringed spaces. And then after that one is hit by a second wall of endless types of morphisms: smooth, flat, finitely presented, etale, etc., etc., etc. One can quickly wonder where the geometry has gone, especially as most algebraic geometry books, for whatever reason, seem to forgo geometric pictures![1]

However, I believe with a bit of category theory, one can get a relatively quick understanding of some of the key basic ideas of the modern viewpoint. In particular, you'll need the following:

- Category theory: understanding of categories, opposite categories, terminal objects, isomorphisms, monomorphisms, epimorphisms, and pullbacks
- Modern algebra: commutative rings, ideals, and quotient rings.

Everything else I'll explain and relate to geometry. And towards the end I'll try to give a sense of how what has been discussed here relates to the (1950's onwards view of) algebraic geometry (the "locally ringed space" approach).

1.1 Acknowledgements

I am enourmously indebted to two algebraic geometry books, which (finally!) helped me start to see a way through algebraic geometry (as well as providing some images which I've borrowed here to illustrate some concepts):

- **The Rising Sea: Foundations of Algebraic Geometry** by Ravi Vakil (available online at http://math.stanford.edu/ vakil/216blog/FOAGnov1817public.pdf)
- **The Geometry of Schemes** by Eisenbud and Harris.

Even though the point of view presented here is different than the one in those textbooks (which follows the standard "locally ringed space" approach), each of these books give lots of intuition and pictures, both of which I think are severely lacking from most standard algebraic geometry books!

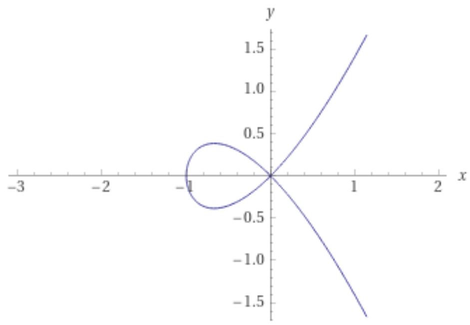
I'd also like to thank Geoff Vooys for very useful discussions, and JS Lemay for (through his insight into differential bundles in cAlg_R^{op}) helping to motivate me to look more closely at algebraic geometry again!

1.2 What is algebraic geometry (classically) about?

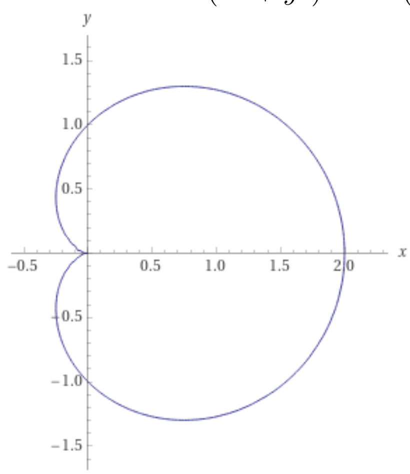
Algebraic geometry studies the zeroes of some set of polynomials over some ring geometrically. Here are a few examples to give you a sense of the variety of things that exist.

The zero set of $y^2 - x^3 - x^2$ in $\mathbb{R}^2$ is the "nodal cubic":

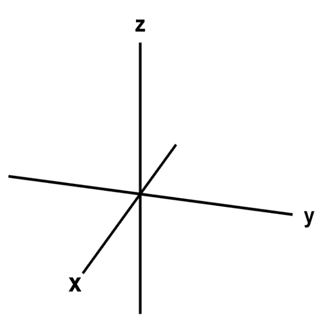
[1] One of my goals in these notes is to have an image representing the geometry of what is going on on almost every page.



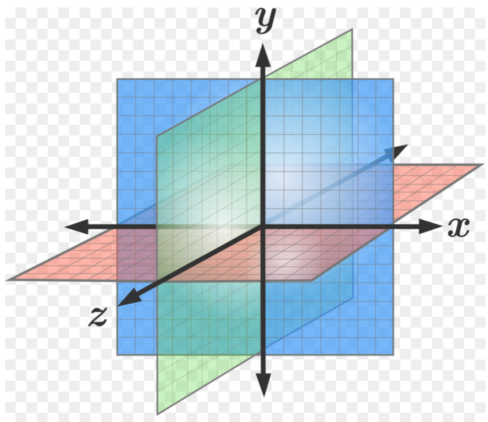
The zero set of $(x^2+y^2)^2 - 2x(x^2+y^2) - y^2$ in $\mathbb{R}^2$ is the “cardioid”:



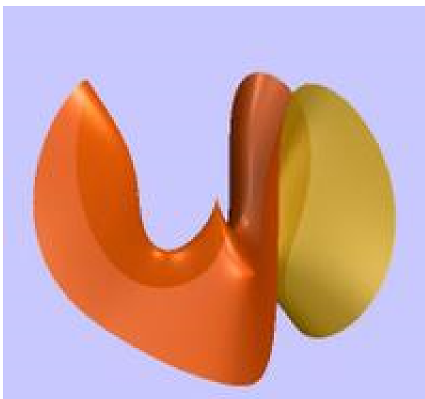
The zero set of the polynomials xy, yz, xz in $\mathbb{R}^3$ is the 3 co-ordinate axes



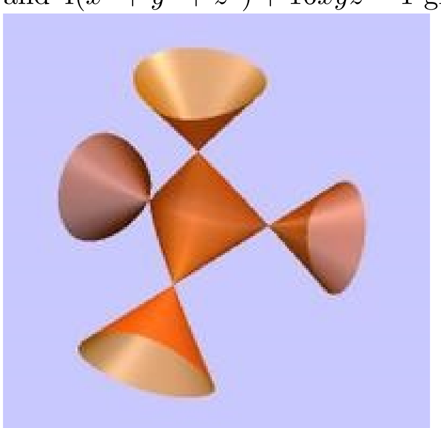
The zero set of the polynomial xyz in $\mathbb{R}^3$ is the 3 co-ordinate planes



We saw the solution set of $xz + y^2(x + y + z)$ above; here are two more of the seemingly endless list of fascinating surfaces that arise: $x^2 + y^2 + z^3 + 3.2(x^3 - 3xy^2)$ gives



and $4(x^2 + y^2 + z^2) + 16xyz - 1$ gives



Note that all of the above examples are "non-smooth": they all contain at least one (and in some cases many) points where the curve/surface intersects itself and/or comes to a sharp point. These types of objects are explicitly ruled out in differential geometry. (By contrast, however, differential geometry allows more maps: in some sense, the maps between algebraic geometry objects are "just polynomials").

If we want to study these objects, then, we have to figure out how to represent them. The classical and most obvious answer is to literally represent these objects as their set of solutions; this is known as the variety associated to the equation (or set of equations). So, the (real) variety of $x^2 - y^3$ is literally the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 - y^3 = 0\}.$$

One can make also make a category of these objects, so that, for example, the unit circle in 2-space and a great circle of a sphere (in 3-space) are isomorphic in that category. However, this category has at least two deficiencies:

1. It doesn't easily allow one to look at the solutions of an equation over different rings.
2. It doesn't keep track of multiplicity, which is important for theorems like the fundamental theorem of algebra and its generalization, Bezout's theorem.

Let's examine each of these in turn.

1.3 Solutions over different rings

As a simple example, consider the equation

$$x^n + y^n = z^n$$

The statement of whether this equation has non-trivial integer solutions for any $n > 2$ is Fermat's last theorem. So even the question of whether a given equation has a solution in a given ring (never mind what they look like) can be very tricky.

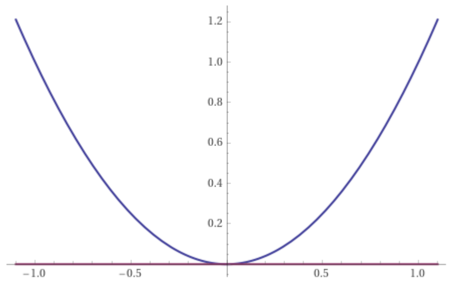
However, a standard technique to try and understand whether solutions exist in some ring R, is to look at whether the equation has solutions in some larger ring $S \supset R$. For example, Falting's theorem states that if one looks at the *complex* solutions of a rational equation have genus ("number of holes") ≥ 1, then there are only finitely many *rational* solutions to the original equation.

So it would be useful to have a setting in which one can look at a given polynomial equation with coefficients in some R, and look not only at its R-solutions, but also its S solutions for any ring $S \supset R$.

1.4 The importance of multiplicity

Bezout's theorem is a *beautiful* result which states that if a curve of degree m and a curve of degree n intersect in finitely many points, then there are exactly mn such points, so long as things are sufficiently nice: (i) the points are in $\mathbb{C}$, (ii) the curves are considered in projective space, (iii) the intersection points are counted with multiplicity.

This last point is the most important for us right now: how does one "keep track of multiplicity"? Unfortunately, the multiplicity information is not tracked if you simply consider the intersection of the associated varieties. For example, even allowing complex and/or projective solutions ("solutions at ∞"), the intersection of $y = x^2$ and $y = 0$ is $x^2 = 0$, so only has one solution, namely $x = 0$:



And so to keep track of multiplicity, we'd like the intersection of $y = x^2$ and $y = 0$ to keep track of the fact that it was an intersection "of multiplicity 2", ie., $x^2 = 0$, not an "intersection of multiplicity 1" like the intersection of $x = y$ and $x = -y$.

2 Affine geometry

Here is where we depart from standard algebraic geometry orthodoxy. To handle the issues noted above, the standard way is to define the category of affine schemes. But this definition has *significant* pedagogical issues. In particular, right off the bat, it starts by associating a topological space to any commutative ring A, where the points of the topological space are the prime ideals of R. *It isn't stressed enough how weird this is*! "Justification" is usually given by Hilbert's Nullstellensatz, which says that the *maximal* ideals of, eg., $\mathbb{C}[x, y]$ correspond to points of the affine complex plane, ie., the set $\mathbb{C} \times \mathbb{C}$ (with a point (w, z) corresponding to the ideal $(x - w)(x - z)$). But why the move to *prime* ideals? And what about non-algebraically closed fields like $\mathbb{R}$, which have the maximal ideal $R[x]/(x^2 + 1)$, which doesn't correspond to a point in the real plane?

The situation only gets worse from there, where one defines a topology on the set of prime ideals which looks very strange (and in most cases of interest all such topologies are homeomorphic to the co-finite topology) and then a sheaf of rings on that topology using localizations. Morphisms are then defined, but then even these require some care, as one has to restrict to morphisms of "locally" ringed spaces, ie., the stalks of the associated sheaves are local rings.

But at the end of day, the category that is built (the category of "affine schemes") **is equivalent to the opposite of the category of commutative rings**! (And, if one is just interested in solutions over some R, instead the opposite of the category of commutative R-algebras). So, the approach we're going to take is just work with this category directly and look at how objects and morphisms in this category can be interpretated geometrically[2]. All these ideas are given in algebraic geometry books, but usually much later than we present here.

2.1 Commutative R-algebras

Recall:

[2]This is also the point of view that Grothendieck advocated for later in his career - see https://webusers.imj-prg.fr/ leila.schneps/grothendieckcircle/FuncAlg.pdf.

Definition 2.1 *For a commutative unitial ring R, a* ***commutative*** *R**-algebra** is a commutative ring A equipped with an action from R satisfy appropriate axioms. Write $cAlg_R$ for the category of commutative R-algebras.*

One can easily check that $cAlg_R$ is equivalent to the coslice category $R/cRing$.

Our main object of interest is the opposite of this category, cAlg_R^{op}.

2.2 Varieties as objects of $\mathbf{cAlg}_R^{op}$

The key point is that we will view a variety defined by some equations $f_1 = 0, f_2 = 0, \ldots f_m = 0$ (in say the variables $x_1, x_2, \ldots x_n$) with solutions in R as an object of $cAlg_R^{op}$ via its associated co-ordinate ring, the quotient ring

$$R[x_1, x_2, \ldots x_n]/(f_1, f_2, \ldots f_n)$$

So for example we view the variety $y^2 - x^2 - x^3 = 0$ over $\mathbb{R}$ as

$$\mathbb{R}[x, y]/(y^2 - x^2 - x^3)$$

or the variety $xy = 0, xz = 0, yz = 0$ over $\mathbb{C}$ as

$$\mathbb{C}[x, y, z]/(xy, xz, yz)$$

In the next few sections, we will give evidence why this is the "right" choice to represent these objects.

2.3 Points in affine schemes

One reason to see why these objects in the category cAlg_R^{op} is "correct" is to consider elements of these objects. Recall that in a category with a terminal object 1, an **element** of an object X is simply a map

$$1 \longrightarrow X$$

while a **generalized element of** X is a map from any object to X.

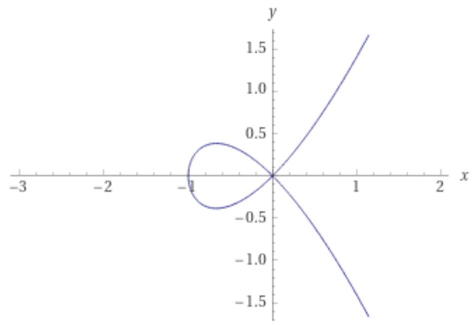
Consider $\mathbb{R}[x, y]/(y^2 - x^2 - x^3)$ as an object of cAlg_R^{op} (with $R = \mathbb{R}$) and consider what its elements are. In this category $\mathbb{R}$ itself is a terminal object, and so an element is a map

$$\mathbb{R} \longrightarrow \mathbb{R}[x, y]/(y^2 - x^2 - x^3)$$

in cAlg_R^{op}, or equivalently a map in cAlg_R

$$\mathbb{R}[x, y]/(y^2 - x^2 - x^3) \longrightarrow \mathbb{R}.$$

Such a map is entirely determined by where it sends x and y, but, given it is modded out by the ideal, it must send x and y to points a and b such that $b^2 - a^2 - a^3 = 0$. In other words, such a map is nothing more than a pair of points (a, b) in $\mathbb{R} \times \mathbb{R}$ which satisfy the equation $b^2 = a^2 + a^3$; that is, it is nothing more than the solutions to the original equation $y^2 = x^3 + x^2$! (Pictured here:)



So the "elements" of this object in cAlg_R^{op} are exactly the sort of thing we want them to be: the solutions to the original equation.

Note that generalized elements are also interesting: by a similar argument to the above, a map $\mathbb{C} \longrightarrow \mathbb{R}[x,y]/(y^2 - x^2 - x^3)$ would exactly be a pair of points in $\mathbb{C}$ which satisfy the required equation (or which there are more).

Another key example is simply the polynomial algebra $R[x_1, x_2, \dots x_n]$. By a similar argument to above, it's elements are exactly n-tuples of elements of R (with no other requirements). Thus, one thinks of the object $R[x_1, x_2, \dots x_n] \in \mathrm{cAlg}_R^{op}$ as "affine n-space (over R)".

Also, note that $R[x]/(x)$ and $R[x]/(x^2)$ both only have one element ($x = 0$). However, they are different rings; $R[x]/(x) \cong R$ while

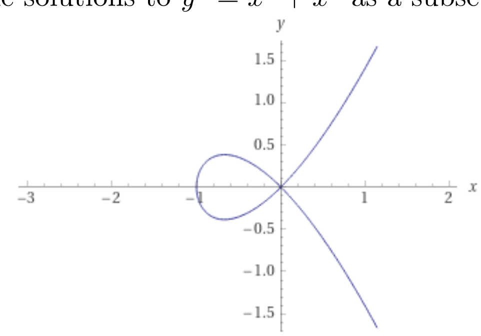
$$R[x]/(x^2) = \{a + bx : a, b \in R, x^2 = 0\}.$$

So not only are we able to represent objects in this category and keep around their important information (their set of solutions), we can also distinguish between the variety represnented by $x = 0$ and the variety represented by $x^2 = 0$ (which, as noted above, is an important consideration for multiplicity).

Note that objects can have no points at all: of course, $\mathbb{R}[x]/(x^2 + 1)$ has no points in cAlg_R^{op} (but does have $\mathbb{C}$ points).

2.4 Subobjects I: Quotients

For the next point in favour of this category, consider that we are geometrically viewing the solutions to $y^2 = x^3 + x^2$ as a subset of 2-space.



(ie. the blue curve is included in the whole plane). Thus, as above, since we are thinking of 2-space as the polynomial ring $R[x, y]$, we expect there should be a *monomorphism*

$$R[x,y]/(y^2 - x^3 - x^2) \hookrightarrow R[x,y]$$

in our "geometric category" cAlg_R^{op} which represents this inclusion. So this should come from a natural *epimorphism*

$$R[x,y] \to R[x,y]/(y^2 - x^3 - x^2)$$

in our algebraic category cAlg_R. But of course there is: the quotient map! So the quotient map (in the algebraic category) becomes the natural inclusion of a variety in n-variables into affine n-space (in the geometric category).

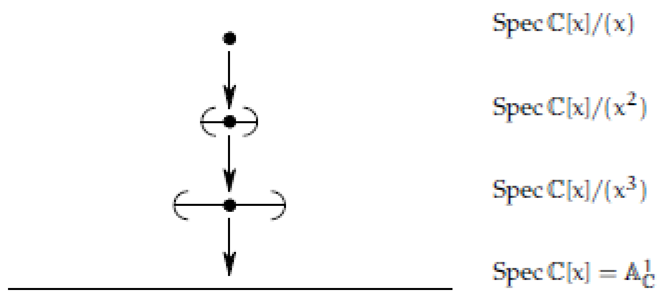
So we naturally have varieties as subobjects of affine n-space. But note that we have other interesting subobjects as well: as above, the subobject

$$R[x]/(x)$$

repesents the point 0, but we also have the subobject

$$R[x]/(x^2)$$

which is different than the mere point 0. One can think of this as an "infinitesimally thickened" point. For example, Vakil (pg. 134) draws it as a point with a bracket around it, and similarly $R[x]/(x^3)$ as a point with a larger bracket around it, etc:



So moving to this category of "affine schemes" has enriched our geometric picture considerably already (and we'll see more in the next section): not only does the affine line or plane contain the ordinary points, it also contains infinitesimally thickened points (and infinitesimally thickened lines, etc., eg., $R[x,y]/(x^2)$ is an infinitesimal thickening of the line $x = 0$ in the plane).

2.5 Subobjects II: Localizations

So, quotients in the algebraic category give important examples of subobjects in the geometric category. But there is another large class of epimorphisms of algebras, hence also give subobjects in the geometric category, and again have meaningful geometric interpretations: localizations. They will geometrically correspond to open subsets, germs, and the fascinating "generic points".

On the algebra side, recall:

Definition 2.2 *If R is a ring, a* ***multiplicative set*** *in A is a subset $S \subseteq A$ which is closed under multiplication and contains 1. The* ***localization*** *of A at S, written $A[S^{-1}]$, is (informally) the ring formed from A by forcing all elements of S to be invertible. More formally, it is the equivalence class of elements written as $\frac{a}{s}$ where $a \in A, s \in S$, with $\frac{a_1}{s_1} \cong \frac{a_2}{s_2}$ if there exists a $t \in S$ so that $t(a_1 s_2) - s_1 a_2) = 0$.*

Example 2.3 *The localization of $\mathbb{Z}$ at the multiplicative set consisting of all non-zero elements is isomorphic to $\mathbb{Q}$.*

Example 2.4 *If S is the powers of 2, $\mathbb{Z}[S^{-1}]$ are the dyadic rationals: the rationals $\frac{a}{b}$ where b is a power of 2.*

KEY FACT: for any multiplicative set $S \subseteq A$, the inclusion

$$A \longrightarrow A[S^{-1}]$$

is an *epimorphism* in the category of rings (and also in the category of R-algebras if A is an R-algebra). (Note that it is also a monomorphism.) If you haven't done this exercise before, it is definately worth doing, as it seems so counterintuitive at first.

What this means for us, however, is that for any S, we have a monomorphism

$$A[S^{-1}] \longrightarrow A$$

in the geometric category, so we can think of $A[S^{-1}]$ as a subobject of A. How can we think of these geometrically?

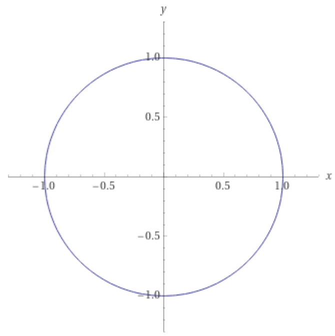
It depends a great deal on the type of multiplicative subset one considers, but all are interesting. There are two main classes of multiplicative subsets of A: for any $a \in A$, the set of all powers of a is a multiplicative subset, and for any prime ideal P of A, the complement of P in A is a multiplicative subset. The localization of A at the complement of P is written as A_P.

- (Open subset example 1) For any element $a \in A$, we can take S to be the powers of a (including 1). In particular, take $k[x]$ (k a field) and S to be the powers of x. What are the k-points of $k[x][S^{-1}]$ in this case? Such a point would be a k-algebra map $k[x][S^{-1}] \longrightarrow k$, so again is entirely determined by where it sends x. But now x has to be something invertible! So its set of points is the "open" subset $\{r \in K : r \neq 0\}$.

- (Open subset example 2) Similarly, if we take S to be all powers of $x^2 + y^2 - 1$ in $k[x, y]$, then a map

  $$k[x, y][S^{-1}] \longrightarrow k$$

  is entirely determined by where it sends x and y, with the sole requirement being that $x^2 + y^2 - 1$ must be invertible. So it is all points that are *not* on the circle (again, an "open" subset):



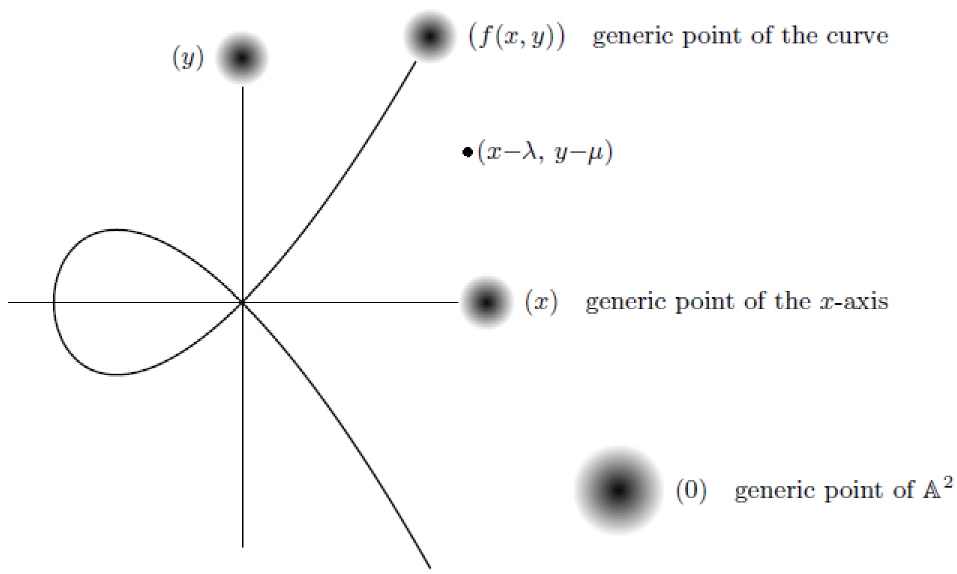
- (Note on open subsets) It is important to note that not all ordinary open subsets of say $\mathbb{R}^n$ can be viewed as rings in this way. For example, there is no ring R whose elements (as an object of cAlg_R^{op}) give the set of non-origin points in $\mathbb{R}^2$ (it was something of a coincidence that one can do this in $\mathbb{R}$). So the "open subsets" one gets from the previous two examples are fairly limited. One way to get more open subsets is to move to the larger category of schemes: for more on this, see Section 4.

- (Generic points 1) If A is an integral domain, then (0) is a prime ideal, and we can form the localization $A_{(0)}$, which is also known as the "field of fractions" of A (all non-zero elements of A are forced to be invertible). *This object has no R-points*! It is a subobject of A, but not the empty set (= initial object in the geometric category = terminal object in the algebraic category = the zero ring). It is called the **generic point of A**, and is quite useful. Miles Reid refers to it as a point "woven into the fabric of A". People draw it geometrically as a fuzzy dot, or a scribble, and it is "everywhere on A but nowhere in particular". We will return to it later.

- (Generic points 2) Note that if a polynomial $p(x, y)$ is irreducible, then (p) is prime, so $A[x, y]/(p)$ is an integral domain, and so each such curve has a generic point, and all of these generic points are subobjects of $A[x, y]$ via the sequence of ring epimorphisms

$$A[x, y] \longrightarrow A[x, y]/(p) \longrightarrow (A[x, y]/(p))_{(0)}$$

So the affine plane contains not only the generic point of the entire plane, but also the generic points of each irreducible curve. (These points, together with the ordinary points of the plane, are what the locally ringed space version of algebraic geometry takes as primitive.). Here's how Eisenbud/Harris pictures generic points:



- (Germs 1) But there's more! Consider the prime ideal (x, y) in $A[x, y]$. How should we geometrically interpret localization at it? In this case the localization $A[x, y]_{(x,y)}$ does still have an A-point, namely $(0, 0)$. But it only has this A-point. However, one can also show that for any irreducible curve p which passes through this point, it's generic point is contained in this localization. So we can think of $A[x, y]_{(x,y)}$ as the germs of smooth curves which pass through the origin. Picture shamelessly copied from Vakil:

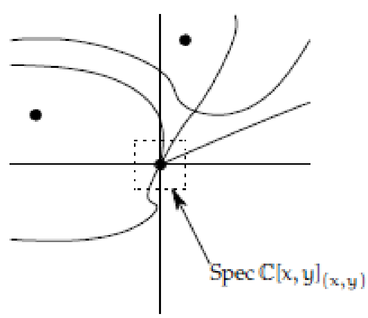


FIGURE 3.5. Picturing Spec $\mathbb{C}[x, y]_{(x,y)}$ as a "shred of $\mathbb{A}^2_{\mathbb{C}}$". Only those points near the origin remain.

- (Germs 2) What happens if localize at the prime ideal corresponding to some irreducible curve, eg., $P = (y^2 - x^2 - x^3)$? In this case the result has no A-points, but does have its generic point, and again the generic points of any curve which passes through that curve. But in this case the only such point is the whole space, so one just has its generic point and the generic point of the whole plane. So it's like one has "concentrated" (localized!) to that curve. Miles Reid talks about how "divisibility theory in this ring is splendid" and notes that in a case like this (a curve in a surface) the ring is a discrete valuation ring (which apparently are handy to calculate with...). But if we go up a dimension and now localize at a curve we can get some additional things, eg., localizing at the circle $x^2 + y^2 = 1$, $z = 0$ we include the generic point of the sphere $x^2 + y^2 + z^2 = 1$, as the "sphere contains that circle". So just like localizing at the point includes all curves through that point.

To sum up some of what has been said above, for an irreducible polynomial $p \in k[x, y]$ (so that (p) is a prime ideal) we have four different constructions we can perform related to it:

1. The quotient $k[x, y]/(p)$ is the curve itself
2. The localization $k[x, y]_p$ (localize at the multiplicative set given as all powers of p) is the open set not containing the curve
3. The localization $k[x, y]_{(p)}$ (localize at the complement of the prime ideal generated by p) really is the "localization" at the curve (concentrate to the curve being a "point", and have as subobjects all larger surfaces, etc., containing that curve)
4. Quotient then localize at 0 or localize then quotient at the ideal are isomorphic (see Vakil pg. 139):
$$(k[x, y]/(p))_{(0)} \cong (k[x, y]_{(p)})/(p)$$
and both represent the generic point of the curve.

Note that if the original p represents a maximal ideal, ie., just a point, then the quotient is already a field, and so the first and fourth items coincide in this case. (The point is its generic point).

2.6 Pullbacks

Pullbacks play a key role in the category of affine schemes, as geometrically they represent several important operations:

- A pullback of two subobjects of a common object represents their intersection.
- Pulling back a "family" of objects parameterized by another object along a point represents an individual member of that family, including the "generic" member (!)
- We can "pull back along an analytic neighborhood" to view local information.

So, let's first see how pullbacks are defined in cAlg_R^{op}. A pullback

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d, "g"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

in cAlg_R^{op} corresponds to a pushout

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

in cAlg_R, and this is given by a tensor product

$$D \cong B \otimes_A C$$

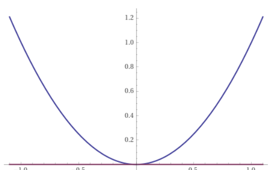
where recall that $\otimes_A$ means that we can move A-scalars across the tensor, ie., for any $a \in A$,

$$f(a)b \otimes c = b \otimes g(a)c$$

in $D \cong B \otimes_A C$.

Here are three key examples:

- (Intersections of varieties) Recall from Bezout's theorem that we'd like the intersection of $y = x^2$ and $y = 0$ to be more than a mere point: it should capture the fact that this intersection "has multiplicity 2".



  So, to verify this, we let A be their intersection; that is, the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k[x,y]/(y - x^2) \arrow[d] \\ k[x,y]/(y) \arrow[r] & k[x,y] \end{tikzcd}$$

  By the above,

$$A \cong k[x,y]/(y) \otimes_{k[x,y]} k[x,y]/(y - x^2).$$

  But the fact that we are tensoring over $k[x,y]$ means that we can move any polynomial between the two factors, so for example

$$0 = y \otimes 1 = 1 \otimes y = 1 \otimes x^2$$

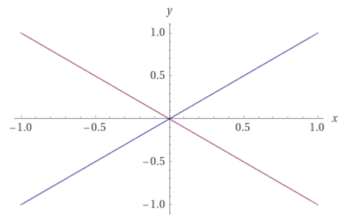
  So in this ring $x^2 = 0$, and we can show

$$A \cong k[x,y]/(y, x^2) \cong k[x]/(x^2)$$

  So indeed we get that their intersection is an "infinitesimally thickened point" - as a variety, it only has one solution ($x = 0$), but as an object of this category, it has kept track of the information that it was formed from $x^2 = 0$. (The multiplicity itself can more precisely be seen as the dimension of A as a vector space over k).

The geometric category cAlg_R^{op} has remembered that "there's more to this intersection than just a point"!

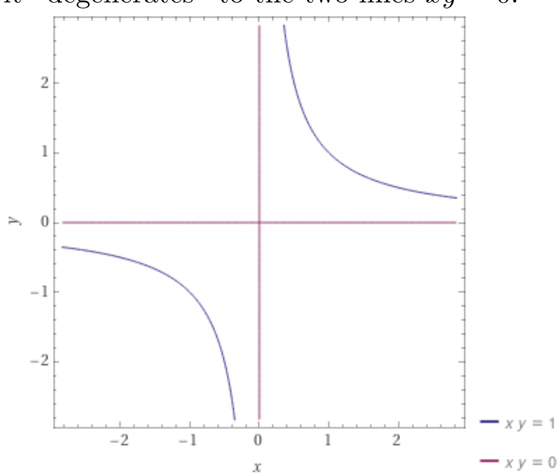
(By contrast, one can check that the intersection of $y = x$ and $y = -x$ really is just a point, ie., their intersection in cAlg_R^{op} is $k[x, y]/(x, y) \cong k$, representing a single point.)



- (Specific fibres of a family) In classical algebraic geometry, one often wants to consider not just a single variety, but a family of varieties parameterized by some value. For example, one might want to consider the family of curves

$$xy = t$$

for varying values of t. For a fixed $t \neq 0$ this looks like a hyperbola, but at $t = 0$ it "degenerates" to the two lines $xy = 0$:



We can represent this family as the canonical monomorphism

$$k[t] \longrightarrow k[x, y, t]/(xy - t)$$

in the algebraic category, which becomes an epimorphism

$$k[x, y, t]/(xy - t) \longrightarrow k[t]$$

in the geometric category. Then we really can think of this as a space fibred over the 1-dimensional line $k[t]$: if we pick a particular point a, represented by giving the algebraic map $k[t] \longrightarrow k$ where $t \mapsto a$, then we can form the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k[x,y,t]/(xy-t) \arrow[d] \\ k \arrow[r, "t \mapsto a"'] & k[t] \end{tikzcd}$$

and one can verify that

$$A \cong k[x,y](xy-a)$$

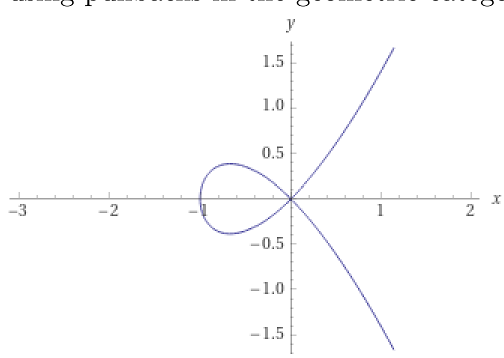
That is, the fibre at $t=0$ really is the two lines above, and the fibre at $t=1$ really is the hyperbola above.

- (Generic fibre of a family) Where this idea gets more interesting, however, is the realization that this can also be done for the "generic point" of the parameterizing space: that is, we can take the inclusion (in the geometric category) of the generic point of $k[t]$, ie., $k(t)$, and take the pullback

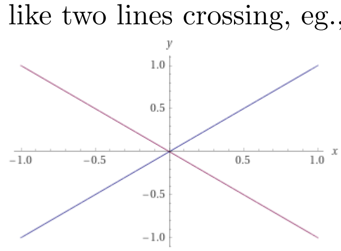
$$\begin{tikzcd} k(t)[x,y]/(xy-t) \arrow[r] \arrow[d] & k[x,y,t]/(xy-t) \arrow[d] \\ k(t) \arrow[r] & k[t] \end{tikzcd}$$

The object $k(t)[x,y](xy-t)$ is then called the "generic fibre" of the family. The idea of it was used classically: one can prove various interesting theorems to the effect of "if something is true about the generic fibre, it is true in the fibre generally, ie., almost everywhere". For example, one can prove the generic fibre is "smooth", representing the idea that almost all fibres of this family are smooth (the only non-smooth one being $t=0$). All this being said, it was only with the advent of the modern approach to algebraic geometry (which allowed one to consider the "generic point" $k(t)$ as an actual object) that one could make precise what one meant by "a generic point" and "considering the generic fibre". And the beautiful thing is that it's just another example of a pullback in cAlg_R^{op}.

- (Analytic neighborhoods and local information) This is my favourite example of using pullbacks in the geometric category. Recall that $y^2-x^2-x^3$ looks like



The interesting point here is the origin, so we would like to understand it. In particular, we would like to say that "near the origin", this object simply looks like two lines crossing, eg., it looks like the origin in $(y-x)(y+x)$:



How can we do this? Recall that we are very limited by the possible "open subsets" in the geometric category; in particular, there is no object which represents, for example, the open set $U = \{(x, y) : x^2 + y^2 < 0.1\}$, so we can't just "zoom into U", ie., take the pullback along U. One choice might be the localization of this curve at the point $(0, 0)$ (ie., the ring given by localizing $R[x, y]/(y^2 - x^2 - x^3)$ at the complement of the prime ideal (x, y)), but as noted earlier, this localization contains "too much global information", and fails to detect how this like curve at the origin looks like two lines crossing.

However, there is a different "local" object we can use instead: the ring of formal power series $R[[x, y]]$. Recall that in this ring, every polynomial with invertible constant coefficient is itself invertible, and one can use this to show that the only element of this ring is the point $x = 0, y = 0$. So it's another ring which has only a single point, but somehow has more information than just that point. In fact, it includes all the "infinitesimally thickened points" eg., $R[x, y]/(x^2, xy, y^2)$, $R[x, y]/(x^3, x^2y, xy^2, y^3)$, and is in a precise sense a "completion" of all these rings.

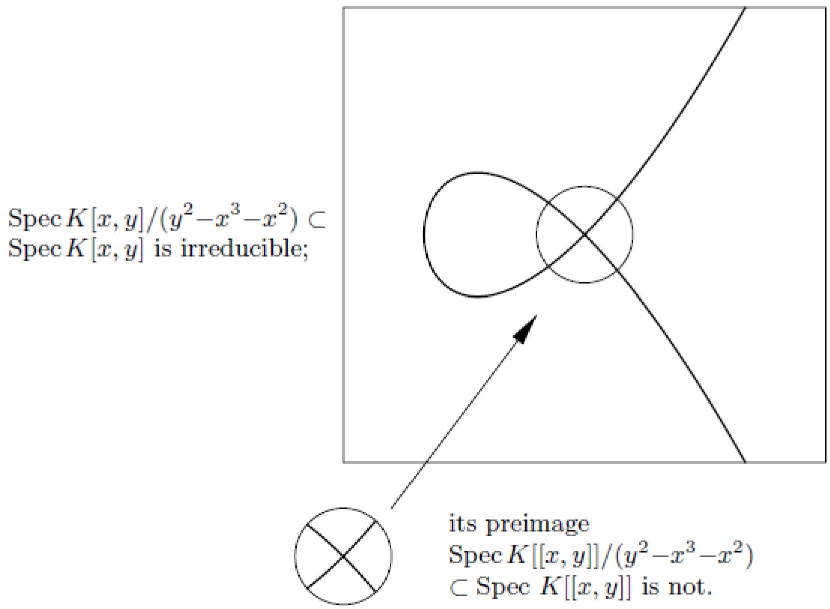
And it is exactly the object we need to investigate the local structure of these kinds of singularities via pullback. That is, we consider the pullback of the variety $y^2 - x^2 - x^3$ along this "analytic neighborhood", ie., we form the pullback

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & R[x,y]/(y^2 - x^2 - x^3) \arrow[d] \\
R[[x,y]] \arrow[r] & R[x,y]
\end{tikzcd}$$

and one can show $A \cong R[[x, y]]/(y^2 - x^2 - x^3)$; one should think of this as an "analytic neighborhood of $y^2 - x^2 - x^3$ at the point $(0, 0)$. And this exactly does what we need: that is, one can show that

$$R[[x, y]]/(y^2 - x^2 - x^3) \cong R[[x, y]]/(y - x)(y + x)$$

making precise the idea that "at the origin, $y^2 - x^2 - x^3$ looks like two crossing lines". From Eisenbud-Harris:

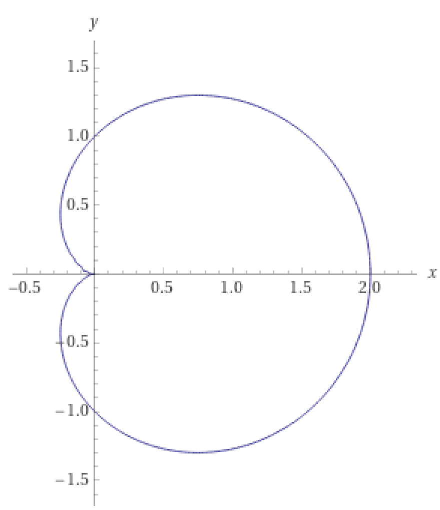


(By the way, why would we get $R[[x,y]]/(y^2-x^2-x^3) \cong R[[x,y]]/(x-y)(x+y)$? This just involves some fiddling around with power series: for example, one can show that if

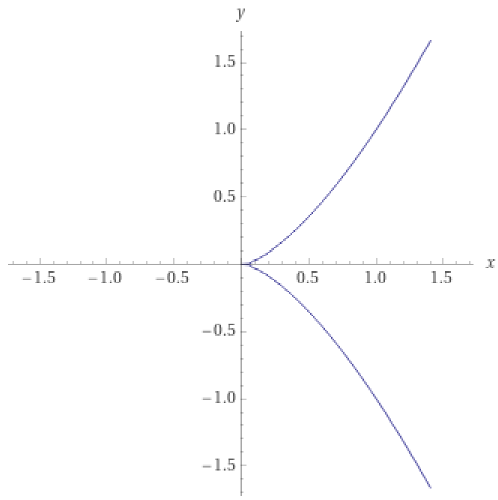
$$u := x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \cdots$$

Then $u^2 = x^2 + x^3$, so that in the power series ring, $y^2 - x^2 - x^3 = y^2 - u^2 = (y-u)(y+u)$. Then one just has to show that the mapping $x \longrightarrow u$ and $y \longrightarrow y$ induces a ring isomorphism $R[[x,y]]/(y^2-x^2-x^3) \cong R[[u,y]]/(y-u)(y+u)$.)

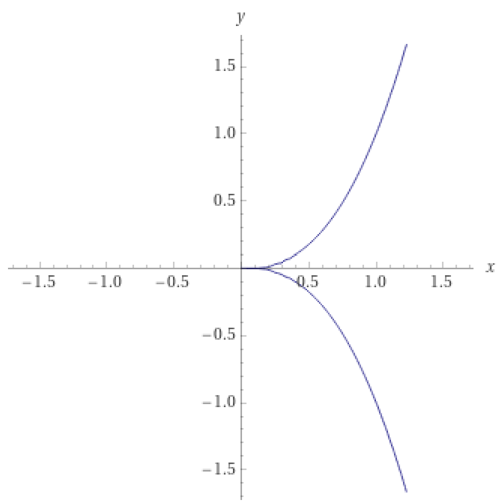
Similar ideas allow one to show that the analytic neighborhoods of the cardioid



and $y^2 - x^3$



at the origin are isomorphic (they both "come into a sharp point"). However, it's worth noting that this notion of "analytic isomorphism" at a point is quite sensitive: for example, $y^2 - x^5$ looks fairly similar to $y^2 - x^3$:

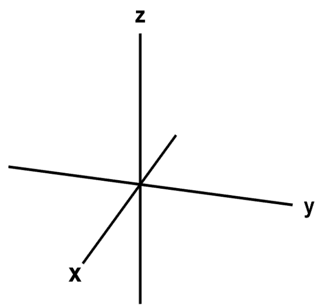


But it is not analytically isomorphic to $y^2 - x^3$ at the origin. So this isomorphism is not only keeping track of roughly what these points look like, but also keeping track of things like "how closely the branches come to touching each other".

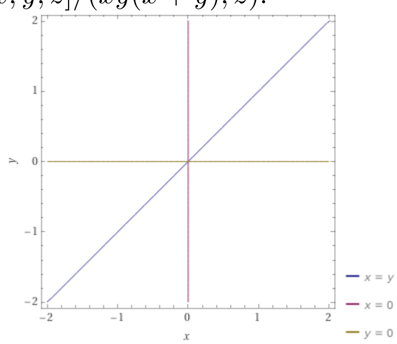
All of these ideas were known in classical algebraic geometry, but a key point here is that the category of affine schemes allows one to view all of these ideas (intersections of varieties, fibres of a family, zooming in on an analytic neighborhood) as examples of a single (categorical) concept: pullbacks.

2.7 Side note: isomorphisms of affine schemes

As an aside, it's worth pointing out something that has tripped me up a few times: the nature of isomorphisms of affine schemes. I'm used to thinking about geometric objects topologically, so that one can move around various bits of the space and still have something isomorphic. This is definately not the case in algebraic geometry. For example, one can have two non-isomorphic varieties consisting of 3 lines with a single common point: the union of the 3 co-ordinate axes, represented by the affine scheme $\mathbb{R}[x, y, z]/(xy, yz, xz)$:



is not isomorphic to 3 lines through the origin in the plane $z = 0$, represented by $R[x, y, z]/(xy(x + y), z)$:



(imagine this in 3-space lying in the plane $z = 0$). Both varieties consist of 3 points through the origin, but they are not isomorphic (see discussion at the end of section 3.3 for a precise proof). This already gives one the sense that isomorphisms in this category are quite subtle.

For another example, there is a famous open question to simply determine the automorphisms of affine n-space, even for $n = 2$ (see the Jacobian conjecture).

2.8 Side note: birational maps

In general, determining if two affine schemes are isomorphic is very hard. One tool that has been developed to help with its study is the notion of a *birational* map. Giving a birational map between two varieties should be thought of as "identifying the two varieties at almost all points". It turns out that giving a birational map between two varieties X and Y is equivalent to giving an isomorphism between their associated generic points (again representing the idea that the generic point of a variety represents its behaviour "almost everywhere").

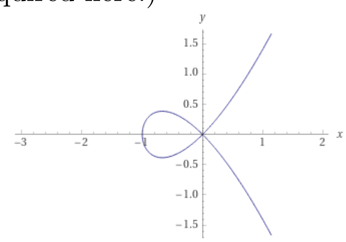
For example, while $k[x, y]/(y^2 - x^3 - x^2)$ and $k[t]$ are certainly not isomorphic (the first has a singularity while the second doesn't) there is a birational map between, that is, an isomorphism of their generic points

$$k(x, y)[y^2 - x^3 - x^2) \longrightarrow k(t)$$

given by sending in the forward direction

$$x \mapsto t^2 - 1, y \mapsto t^3 - t$$

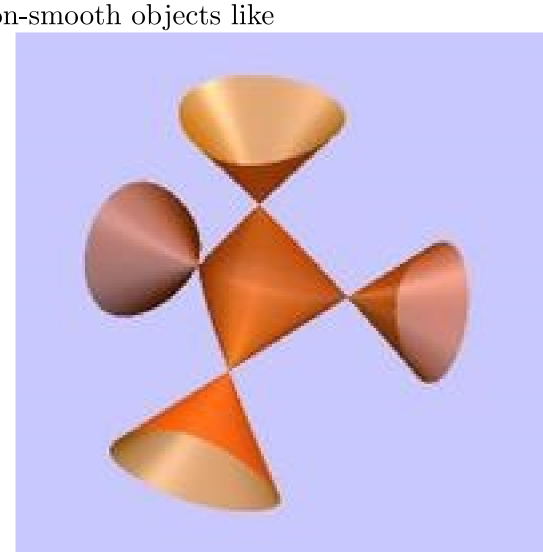
and in the backward direction $t \mapsto \frac{y}{x}$. (It's not a bad idea to check all the details required here!)



An important theorem to show how useful birational maps is **Hironaka's theorem**, which shows that every variety over a field of characteristic 0 is birationally equivalent to a smooth variety (the general for non-zero characteristic is an open problem!) In a sense, then, this reduces the question of isomorphism of such varieties to two separate questions: identifying isomorphism classes of smooth varieties, and identifying different types of singularities that can exist on the non-smooth points.

3 Differentiation and tangents

I find this remarkable: even though algebraic geometry includes the study of highly non-smooth objects like



there is still a well-defined tangent bundle associated to any (affine) scheme, and that satisfies the same axioms as the tangent bundle for smooth manifolds. This section describes this tangent bundle.

3.1 Kahler differentials

The main construction we need on the algebra side is the module of Kahler differentials.

Definition 3.1 *Given an R-algebra A, its* ***module of Kahler differentials*** *is an A-module written as $\Omega_{A/R}$, and defined as follows:*

- *It is the free A-module on the symbols da (one for each $a \in A$) subject to the following relations:*
- *These d's preseve the R-module structure of A directly; that is, for any $a, b \in A$ and $r \in R$,*
$$d(a+b) = da + db, d0 = 0, d(ra) = rd(a)$$
- *But the ring structure of A is not preserved directly; instead, it is preserved "differentialy"; that is,*
$$d(ab) = adb + bda \text{ and } d(1) = 0$$
ie., "d satisfies the Leibniz rule".

This can look very abstract! But it becomes a bit clearer when you look at examples:

- Let's start by looking at $R[x]$. In theory, there is a generator $dp(x)$ for every $p(x) \in R[x]$. But the relations quickly reduce this: for example, by the Leibniz rule,
$$d(x^2) = d(x \cdot x) = xdx + xdx = 2xdx.$$
Similarly $d(x^n) = nx^{n-1}dx$. The module preservation then shows that all $dp(x)$ reduce to some linear combination of dx. So the Kahler differentials of $R[x]$ over R are simply the free R-module on **one** generator (namely, dx).
- Similarly, $\Omega_{R[x_1, \ldots x_n]/R}$ is the free R-module on n generators $(dx_1 \ldots dx_n)$.
- To handle quotients of polynomial rings, one simply differentiates the generating equation(s)! For example, if
$$A = \mathbb{R}[x, y]/(y^2 - x^2 - x^3)$$
then $\Omega_{A/R}$ is the free A-module on the generators dx, dy, subject to the relation
$$d(y^2 - x^2 - x^3) = 0$$
or
$$2ydy - 2xdx - 3x^2dx = 0.$$
- Similarly the Kalhler differentials of $A = \mathbb{R}[x, y]/(xy)$ is the free A-module on dx, dy subject to the relation
$$xdy + ydx = 0.$$
- Some things can be entirely trivial: $\Omega_{R/R} = 0$ (that is, the 0-module) since any r in $R = r \cdot 1$, and $d1 = 0$.

- But you can also have $\Omega_{A/R} = 0$ for $A \neq R$: for example, I claim that $\Omega_{\mathbb{C}/\mathbb{R}} = 0$. Indeed, every $z \in \mathbb{C}$ is of the form $a + bi$ for $a, b \in \mathbb{R}$, so it suffices to know what di is. But since $i^2 = -1$,

$$0 = d(-1) = d(i^2) = 2idi,$$

so $di = 0$. So $\Omega_{\mathbb{C}/\mathbb{R}} = 0$.

- You can also have things that look like they should have 0 Kahler differential not be so. For example, note that if $A = \mathbb{R}[x]/x^2$, A has only one point: $x = 0$. But it's Kahler differentials are non-zero: they are the free A-module generated by dx subject to the relation

$$2xdx = 0$$

and this doesn't force $dx = 0$, it just means that $\Omega_{A/\mathbb{R}}$ is the free $\mathbb{R}$ module on the generator dx (ie., all its terms are of the form rdx for $r \in \mathbb{R}$).

3.2 Symmetric algebra and the tangent bundle

Suppose A is an R-algebra. Then the **symmetric** A **algebra** of an A-module M (written as $\mathrm{Sym}_A(M)$) is the free commutative A-algebra generated by M: that is, it has:

- terms of the form $a \cdot 1$ for $a \in A$ (these are usually just written as a)
- terms of the form $am_1m_2 \dots m_n$ for $a \in A$ and $m_1, m_2, \dots m_n \in M$
- and A-linear combinations of these terms.

Again, this can look complicated but is easier to understand on examples:

- If M is the free A-module on one generator t, then $\mathrm{Sym}_A(M)$ is simply the polynomial ring $A[t]$.
- Similarly, if M is the free A-module on n generators $t_1, t_2, \dots t_n$, then $\mathrm{Sym}_A(M)$ is simply the polynomial ring $A[t_1, \dots t_n]$.
- Any relations are preserved by this process, eg., if M is a free module quotiented by some relation(s), then $\mathrm{Sym}_A(M)$ will by the polynomial ring quotiented by those same relations.

To build the tangent bundle of an R-algebra A, we perform these two constructions in turn: so the tangent bundle of A will be the symmetric algebra of the Kahler differentials of A, and we will write this as T_RA. For example:

- If $A = R[x]$, then $\Omega_{A/\mathbb{R}}$ is the free A-module on the generator dx, so

$$T_R(A) = R[x][dx] = R[x, dx]$$

ie., the polynomial ring on 2 variables.

- Similarly, if $A = R[x_1, \dots x_n]$,

$$T_RA = R[x_1, \dots x_n, dx_1, \dots dx_n]$$

This is the analogue of (in smooth manifolds) for a Euclidean space $\mathbb{R}^n$, $T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$!

- By our discussion of generators and relations above, it's also easy to calculate the tangent bundle of varieties. For example, $A = \mathbb{R}[x, y]/(y^2 - x^2 - x^3)$, then

$$T_{\mathbb{R}}A = \mathbb{R}[x, y, dx, dy]/(y^2 - x^2 - x^3, 2ydy - 2xdx - 3x^2dx)$$

- Similarly, if $A = \mathbb{R}[x, y]/(y^2 + x^2 - 1)$, then

$$TA = \mathbb{R}[x, y, dx, dy]/(y^2 + x^2 - 1, 2ydy + 2xdx)$$

- For $\mathbb{C}$ seen as an $\mathbb{R}$-algebra,

$$T_{\mathbb{R}}(\mathbb{C}) = \mathbb{C}.$$

  Note how odd this last example is! The only smooth manifolds whose tangent bundle equal themselves are finite copies of the 0-dimensional manifold, but that's not the case here.

For another class of examples, it is easy to calculate the tangent bundle of a localization: for any multiplicative set U of a ring A, the following is a pullback:

$$\begin{tikzcd}
T(A[U^{-1}]) \arrow[r] \arrow[d] & TA \arrow[d] \\
A[U^{-1}] \arrow[r] & A
\end{tikzcd}$$

(where the maps down are the canonical algebra inclusion $X \to TX$). In particular, for any $f \in A$, recall that localizing A at the multiplicative set generated by A, A_f, can be thought of as the "open set not including the variety defined by f". In this particular case we have

$$T(A_f) = [T(A)]_f$$

3.3 Tangent spaces

But how is this construction actually a "tangent bundle"? The next section will describe how it satisfies the axioms of a tangent category, but before we get there, I think it's useful to get some geometric intuition for this "tangent bundle". And one way to do this is by looking at the fibres of the tangent bundle of A at a point X of A.

Call this fibre the **tangent space of A at x**, and it is defined to be the pullback

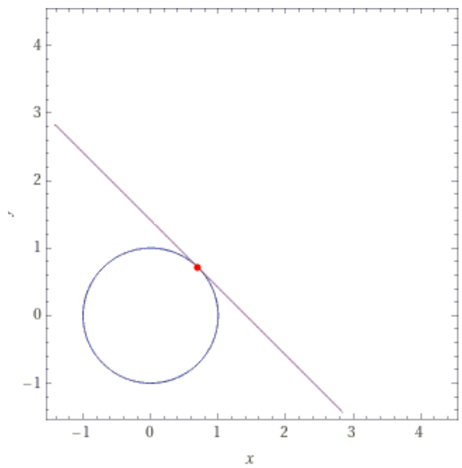
$$\begin{tikzcd}
T_xA \arrow[r] \arrow[d] & TA \arrow[d, "p_M"] \\
1 \arrow[r, "x"'] & M
\end{tikzcd}$$

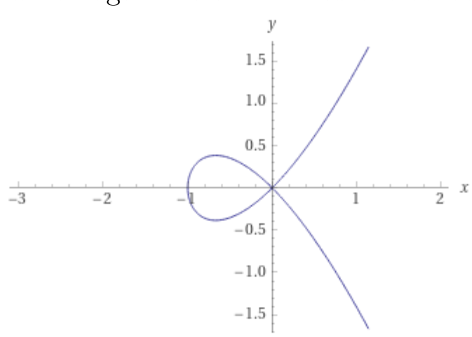
As noted earlier, this pullback in cAlg_R^{op} becomes a pushout in cAlg_R:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "x"'] & TA \arrow[d] \\
R \arrow[r] & TA \otimes_A \mathbb{Z} = T_xA
\end{tikzcd}$$

and one can show that this pushout is simply TA where we apply m to elements of A.

So, let's consider what happens when we take the tangent space of some standard varieties!

- Let $A = \mathbb{R}[x, y]/(x^2+y^2-1)$ (ie., the circle), and consider the point $z = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. As above
$$TA = \mathbb{R}[x, y, dx, dy]/(y^2 + x^2 - 1, 2ydy + 2xdx)$$
but in the tangent space we force $x = y = \frac{1}{\sqrt{2}}$, making this
$$\mathbb{R}[dx, dy]/(\sqrt{2}dy + \sqrt{2}dx)$$
So in this algebra $dy = -dx$. So it's entirely generated by either dx or dy, so simply the one-dimensional polynomial ring $\mathbb{R}[t]$. And note that the relation is exactly the right geometric relation for this point on the circle!


- Now let's consider the more interesting example
$$A = \mathbb{R}[x, y]/(y^2 - x^2 - x^3)$$
which again we think of as the curve


Then as above
$$T_{\mathbb{R}}A = \mathbb{R}[x, y, dx, dy]/(y^2 - x^2 - x^3, 2ydy - 2xdx - 3x^2dx)$$

and if take the tangent space at the left-most point $x = 1, y = 0$ we get

$$\mathbb{R}[dx, dy](2dx - 3dx)$$

so in this algebra $-dx = 0$, so $dx = 0$ and dy is free. So again it's one-dimensional (and again the geometric picture is right: no change in the x-direction, free change in the y-direction).

- But what happens if we take $x = 0, y = 0$? Then the relation

  $$2ydy - 2xdx - 3x^2dx$$

  simply becomes 0! So in this case the tangent space is completely free on dx and dy, ie.,

  $$T_{(0,0)}(A) = \mathbb{R}[dx, dy]$$

  So this notion of tangent space is finding that the tangent space at the origin should be 2-dimensional! In other words, it is detecting the singularity at this point but how its tangent space changes dimension there.

- Something similarly interesting happens with the "infinitesimally thickened point"

  $$A = \mathbb{R}[x]/(x^2)$$

  As noted above, this only has one point: $x = 0$. But if we apply this to the tangent bundle

  $$TA = \mathbb{R}[x, dx]/(x^2, 2xdx)$$

  Again we get no relation, just the free module $\mathbb{R}[x]$. **So even though this A has "only one point", its tangent space at that point is non-trivial!**

Another use for tangent spaces is to show that two affine schemes are not isomorphic. Recall the example from Section 2.7: we would like to show that the three co-ordinate axes

$$A_1 := R[x, y, z]/(xy, yz, xz)$$

and three lines in the plane $z = 0$

$$A_2 := R[x, y, z]/(xy(x + y), z) = R[x, y, z]/(x^2y + y^2x, z)$$

are not isomorphic. It's easy to check that in both these affine schemes, at any point but the origin, the tangent space is dimension 1(in particular, isomorphic to $R[x]$). However,

$$TA_1 = R[x, y, z, dx, dy, dz]/(xdy + ydx, ydz + zdy, xdz + zdx)$$

so

$$(TA_1)_{(0,0,0)} = R[dx, dy, dz]$$

But

$$TA_2 = R[x, y, z]/[x^2dy + 2xydx + y^2dx + 2xydy, dz)$$

So

$$(TA_2)(0, 0, 0) = R[dx, dy, dz]/(dz) = R[dx, dy]$$

So the tangent spaces have different dimension: one is 3-dimensional, the other 2-dimensional. (Which makes sense since in "the three lines in A_2 are in the same plane").

3.4 Tangent category structure

Tangent categories are an axiomatization of a "category equipped with a tangent bundle". The standard example is the category of smooth manifolds. But intriguingly, the tangent bundle described in this section is also a tangent category. This section describes this structure.

First, note that T is a functor, whose action on morphisms is straightforward: for $f : A \to B$, $T(f) : TA \to TB$ is given by sending

$$a \mapsto f(a), da \mapsto d(f(a)).$$

For the most part, the rest of the structure is a "reflection" of the tangent structure on Cartesian spaces. I'll describe them as maps in $cAlg$:

- $p : A \to TA$ is defined by $a \mapsto a$
- $0 : TA \to A$ is

$$a \mapsto a$$

$$da \mapsto 0$$

- Note that just as TA has generators a, da, T^2A has generators $a, da, d'a, d'da$, and so we define $\ell : T^2A \to TA$ as

$$a \mapsto a$$

$$da \mapsto 0$$

$$d'a \mapsto 0,$$

$$d'da \mapsto da$$

- $c : T^2A \to T^2A$ is

$$a \mapsto a$$

$$da \mapsto d'a$$

$$d'a \mapsto da,$$

$$d'da \mapsto d'da$$

- Summation is a bit wacky if you haven't seen the comultiplication on the symmetric algebra. It's a map $+ : TA \to T_2A = TA \otimes_A TA$, and is defined by

$$a \mapsto a \otimes a$$

$$da \mapsto da \otimes 1 + 1 \otimes da.$$

Also: the Kahler differentials and the symmetric algebra construction both have universal properties, and these universal properties can be used to show that this tangent category structure is representable, with associated "infinitesimal object" the "thickened point" discussed above:

$$R[x]/(x^2)$$

3.5 Summing up geometric vs algebraic constructions

As noted in the introduction, we've built the following table:

Geometric object	Algebraic object
Affine n-space	Polynomial ring $R[x_1, \dots, x_n]$
Variety over R	Quotient ring $R[x_1, \dots, x_n]/I$
Solution of X in R	Ring morphism $X \to R$
Open subset in X not including f	Localization of X at f
Germ at variety represented by P	Localization of X at prime ideal P
Generic point of variety represented P	Residue field $k(P)$ of prime ideal P
Analytic neighborhood at origin	Formal power series ring $R[[x_1, \dots x_n]]$
Pullback (eg., intersection, fibre...)	Tensor product $B \otimes_A C$
Tangent bundle	Symmetric algebra of Kahler differentials

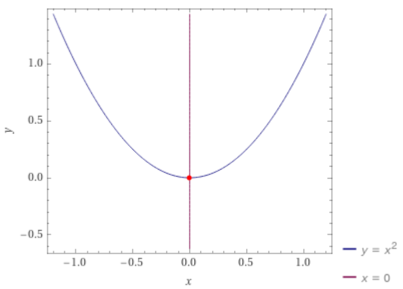
4 Projective geometry

The previous section gave us some understanding of constructions in the geometric category cAlg_R^{op} (= "affine schemes") and how it relates to geometry. In this section we want to take the next step and see how and why one would want to move from affine schemes to schemes.

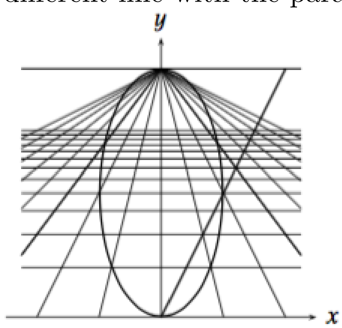
In most books, the following analogy is presented: as Cartesian spaces are to smooth manifolds, so affine schemes are to schemes. That is, just as smooth manifolds "glued together bits of Cartesisan spaces", so one should think of schemes as "glued together bits of affine schemes". But for a while, I couldn't understand *why* one would want to do this. It seemed clear to me why one would want to move from open subsets of Cartesian space to smooth manifolds, as most smooth spaces of interest cannot be represented as open subsets of a Cartesian space. So, to get anything interesting at all (eg., a circle, sphere, or torus) one has to glue together bits of Cartesian spaces. And representing smooth objects in this way means that locally one can work as if one is in Cartesian space.

But why is this needed in algebraic geometry? We already have many interesting spaces in the category of affine schemes alone (eg., see the pictures in the introduction). Why do we need to glue such spaces together?

The main answer is projective geometry. Recall that one of our motivations for the move from varieties to the "geometric category" cAlg_R^{op} = affine schemes was the need to keep track of multiplicity of intersections, and the motivation for this came from Bezout's theorem. But Bezout's theorem is also only valid if we work in *projective* space, not just affine space. For example, in affine space, the intersection of the 2nd degree equation $y = x^2$ and the first degree equation $x = 0$ has only one intersection point:



But in projective space, all parallel lines meet at a "point at infinity", and the line $x = 0$ meets the parabola at such a point at infinity (in addition to their intersection at the origin), as you can see in this image (which also shows the two intersection points of a different line with the parabola):



So representing projective space and the related projective varieties ("zeroes of (homogenous) polynomials in projective space") is important for algebraic geometry.

Unfortunately, projective space and projective varieties cannot be handled in the geometric category cAlg_R^{op} = affine schemes. But one can view projective space and projective varieties as "glued together affine schemes". So our main purpose in this section is to quickly describe how to work with projective geometry, then see how one could view projective space and projective varieties as glued together affine schemes.

4.1 Projective geometry I: homogenous co-ordinates

We'll start by focusing on how to think about the projective plane. One can think of this as an "enhanced" version of the affine plane: it includes a copy of the ordinary affine plane, but also a "point at infinity" for each set of parallel lines in the plane (and all parallel lines will meet at this point). For example, there will be a point where all lines with slope 1 meet, a different point where all lines with slope 2 meet, a different line where all horizontal lines (lines with slope 0) meet, and a different point where all vertical lines (lines with "slope ∞") meet.

How can we represent this more precisely? One way is via homogenous co-ordinates. A **homogenous co-ordinate** for a point in the projective plane is an equivalence class of 3-tuples

$$(x_0, x_1, x_2)$$

where at least one of x_0, x_1, x_2 is non-zero, and for any $\lambda \neq 0$,

$$(x_0, x_1, x_2) \equiv (\lambda x_0, \lambda x_1, \lambda x_2).$$

How does this relate to the description above? Any homogenous co-ordinate with $x_2 \neq 0$ has a unique representative where $x_2 = 1$ (the point $(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)$), and these points we think of as those in the ordinary affine plane - for example, the point $(2, 3)$ in the affine plane corresponds to the (equivalence class of) the homogenous co-ordinate $(2, 3, 1)$; the origin $(0, 0)$ corresponds to the homogenous co-ordinate $(0, 0, 1)$, etc.

The extra "points at infinity" are the points where $x_2 = 0$, where we think of $(a, b, 0)$ as the point at infinity where all lines with slope $\frac{b}{a}$ meet. So, $(1, 1, 0)$ is the point where the lines with slope 1 meet, $(1, 0, 0)$ is the point where all horizontal lines (slope 0) meet, and $(0, 1, 0)$ is the point where all vertical lines meet. (And note that for such points, the equivalence relation forces all that matters to be the ratio of the first two co-ordinates, eg., $(1, 1, 0) \equiv (2, 2, 0)$, $(1, 0, 0) \equiv (-1, 0, 0)$, etc. So it really is just capturing one point for each possible set of parallel lines.)

It is impossible to represent the projective plane in even 3 dimensions (it is possible in 4 dimensions), so as a whole, it is quite hard to visualize. But one can visualize "slices" of it fairly readily: the image above corresponds to viewing the projective plane, by standing at the origin, looking straight down the axis $x = 0$, and seeing all vertical lines meet a single point. From this vantage point one can't see the other parallel lines, but if you look to the left or right, each angle you turn your hand at one can see the other such parallel lines meeting at infinity (for example, turning you head 45 degrees to the right you'd see all the lines with slope 1 meeting).

4.2 Projective geometry II: Projective varieties

What about varieties in projective space? It turns out that one can still look at the "zero sets of polynomials" in projective space, *so long as the polynomials are homogenous.* That is, for a homogenous polynomial like

$$xy^2 - z^3$$

(x, y, z) evaluates that polynomial to 0 if and only if for any $\lambda \neq 0$, $(\lambda x, \lambda y, \lambda z)$ evaluates that polynomial to 0, so the zero set of a homogenous polynomial in projective space is a well-defined concept.

But even better, given any polynomial we might be interested in affine space, say $y - x^3$, we can make it into a homogenous polynomial in projective space by adding sufficient z terms to equal the highest degree term in the original. Using this process,

$$x - y^3 \text{ would become } xz^2 - y^3$$

and

$$xy - x + x^3y^4 \text{ would become } xyz^5 - xz^6 - x^3y^4.$$

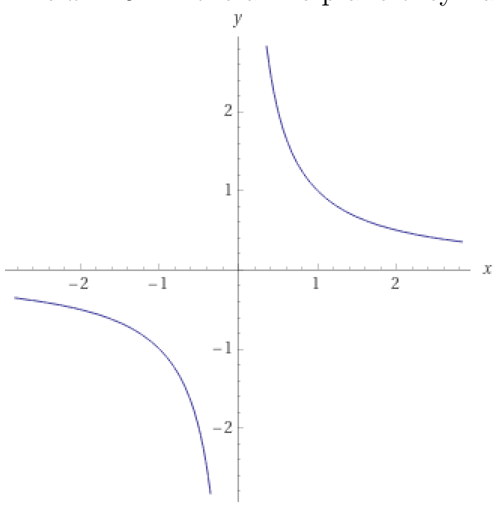
Note that we haven't changed the solutions in the affine plane: if we set $z = 1$, we get the original equation.

But by doing this one can actually calculate all the intersections guaranteed by Bezout's theorem! For example, Bezout's theorem says that the parabola $y - x^2 = 0$ should have two intersection points with $x = 0$ in the projective plane. We can verify this by converting $y - x^2$ into the homogenous polynomial $yz - x^2$ ($x = 0$ is already homogenous) and then taking their intersection in the projective plane: this forces

$$yz = 0$$

so $y = 0$ or $z = 0$. The point $y = 0$ corresponds to the origin in the plane $(0, 0, 1)$, while the point $z = 0$ corresponds to the "point at infinity" $(0, 1, 0)$.

For another example, let's find the intersection points of the hyperbola $xy = 1$ with the line $x = 0$. In the affine plane they have no intersection.



But projectively, we convert $xy = 1$ to $xy = z^2$, and then intersect with $y = 0$ to get $z^2 = 0$. So there is only a single point of intersection, at the point at infinity $(1, 0, 0)$ - but that intersection has multiplicity 2! You can even get a sense of this from the image above, with both branches of the hyperbola coming to touch the point at infinity where all horizontal lines meet.

This is really just verifying Bezout's theorem in a few small examples, but I found it helpful to actually do these calculations to "see" these intersections happening.

4.3 Projective varieties as glued-together affine varieties

As mentioned above, unfortunately there is no way to represent projective space and projective varieties in the category of affine schemes cAlg_R^{op}. But in this section I'll describe how we can view such objects as "glued together" affine schemes.

In particular, let's consider the projective variety

$$x_0 x_1^2 - x_2^3 = 0$$

We want to find its solutions in the projective plane, so we are are looking for (equivalence classes of) projective co-ordinates (x_0, x_1, x_2) which satisfy the equation above. Recalling that we must have at least one of x_0, x_1, x_2 non-zero, we can then split finding solutions into three cases:

- If $x_0 \neq 0$, then by dividing by x_0^3, we can rewrite the equation as

$$\left(\frac{x_1}{x_0}\right)^2 - \left(\frac{x_2}{x_0}\right)^3 = 0$$

Thus by making the change of co-ordinates $x_{1,0} := \frac{x_1}{x_0}, x_{2,0} := \frac{x_2}{x_0}$, we get that any such solution is equivalent to finding a solution to the equation

$$x_{1,0}^2 - x_{2,0}^3 = 0$$

in the ***affine*** plane $\mathbb{R}[x_{1,0}, x_{2,0}]$. Call this corresponding affine scheme A_0; that is,

$$A_0 := \mathbb{R}[x_{1,0}, x_{2,0}]/(x_{1,0}^2 - x_{2,0}^3)$$

- Similarly, if $x_1 \neq 0$, then a solution to the original projective variety is equivalent to finding a solution to

$$x_{0,1} + x_{2,1}^3 = 0$$

in the affine plane (after making the change of co-ordinates $x_{0,1} := \frac{x_0}{x_1}$, $x_{2,1} := \frac{x_2}{x_1}$). Call the corresponding affine scheme A_1.

- And if $x_2 \neq 0$, then any such solution is equivalent to finding a solution to

$$x_{0,2}x_{1,2}^2 - 1 = 0$$

in the affine plane (after making the change of co-ordinates $x_{0,2} := \frac{x_0}{x_2}$, $x_{1,2} := \frac{x_1}{x_2}$). Call the corresponding affine scheme A_2.

So every point on the original projective variety is a solution to at least one of the three affine varieties above. **But we have to be careful**: many solutions will appear in more than one of the three affine varieties: for example, any solution with $x_0, x_1 \neq 0$ will appear in both A_0 and A_1, and any solution with $x_0, x_1, x_2 \neq 0$ will appear in A_0, A_1, and A_2!

So wanting each solution to appear only once forces us to "glue" together the above affine varieties along the common intersections. For example, we want to glue A_0 and A_1 along their common subset, which is where x_0 and x_1 are both non-zero. In A_0 we have already forced x_0 to be non-zero, so we want to look at its solutions where $x_1 \neq 0$ as well. And since we are thinking of $x_{1,0}$ as $\frac{x_1}{x_0}$, this is equivalent to asking that $x_{1,0}$ be non-zero.

Key point: **we know how to look at the subset of points of an affine variety where terms are non-zero, via localization!** That is, the subset of A_0 where $x_1 \neq 0$, that is, $x_{1,0} \neq 0$, is represented by the affine scheme

$$[A_0]_{x_{1,0}} = [R[x_{1,0}, x_{2,0}]/(x_{1,0}^2 - x_{2,0}^3)]_{x_{1,0}}$$

Similarly, the points of A_1 where $x_0 \neq 0$ are represented by the localization of A_1 where $x_{0,1} \neq 0$:

$$[A_1]_{x_{0,1}} = [R[x_{0,1}, x_{2,1}]/(x_{0,1} - x_{2,1}^3)]_{x_{0,1}}$$

And these affine schemes are isomorphic, via the isomorphism

$$[A_0]_{x_{1,0}} \longrightarrow [A_1]_{x_{0,1}}$$

which maps

$$x_{1,0} \mapsto (x_{0,1})^{-1}, x_{2,0} \mapsto (x_{0,1})^{-1}x_{2,1}$$

(Recall that by definition localizing at a point makes that point invertible, so these are well-defined).

Similarly, we can show that "the part of A_1 where x_2 is non-zero" and "the part of A_2 where x_1 is non-zero" are isomorphic, and similarly for A_0 and A_2.

Thus, we have the idea that by specifying the affine schemes A_1, A_2, A_3, along with the "gluing data" of how certain "open subsets" of each pair of affine schemes glues together, we are specifiying the data whose points are those of the projective variety

$$x_0 x_1^2 - x_2^3 = 0$$

The same idea works for any projective variety, even projective n-space itself: for example, we can specify projective 2-space by giving three copies of affine 2-space glued together as above. Projective 1-space is given by two copies of affine 1-space, say $R[x_{1,0}]$ and $R[x_{0,1}]$, with

$$R[x_{1,0}]_{x_{1,0}} \cong R[x_{0,1}]_{x_{0,1}}$$

via the map $x_{1,0} \mapsto (x_{0,1})^{-1}$. We can think of this pictorially as the following gluing (from Vakil's notes, pg. 142):

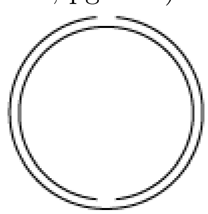


FIGURE 4.7. Gluing two affine lines together to get $\mathbb{P}^1$

(So the projective line is like a circle - but note that the projective plane is not the sphere).

4.4 Aside: tangent bundle of glued-together affine schemes

One can extend the tangent bundle of the previous section to spaces which are "glued together affine schemes" by defining such a space's tangent bundle to be the result of gluing together those affine schemes' tangent bundles.

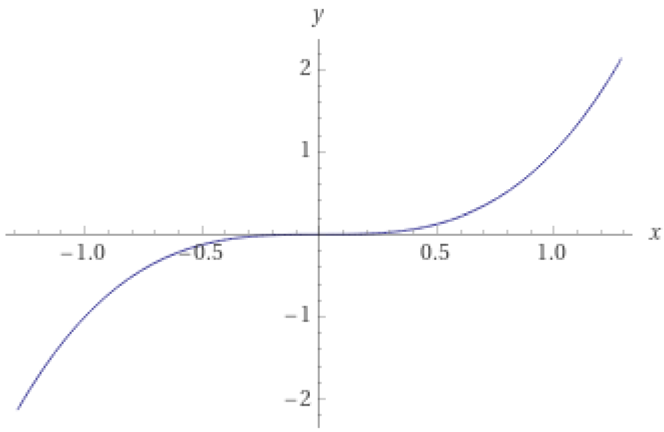
Here's an example, with a small surprise: the process of homogenizing a variety (as described in the previous section) can introduce singularities! For example, in affine space the cubic $y = x^3$: all its tangent spaces are dimension 1. But if we "projectivize" as above, we get the projective variety

$$yz^2 = x^3$$

If we then focus on the "chart" given by setting $y = 1$, we get the affine variety

$$z^2 = x^3$$

which has a singularity at the origin! Translated back to the projective variety, this tells us there is a singularity at the "point at ∞" $(0, 1, 0)$. One can actually geometrically see this by drawing $y^2 = x^3$



"projectively" (as we did earlier with the parabola); if you do this carefully, you'll see you get a cusp at the "point at ∞" $(0,1,0)$.

4.5 Making the "gluing" idea precise

From the previous section, we saw how to think of a projective variety as "a bunch of affine varieties glued together". Thus, the category of schemes is "objects which are glued together bits of affine varieties", and the above construction shows how to define a scheme from any projective variety.

I won't go into how to make this "gluing" idea precise. But there are at least three possibilities on how to do so:

- Standard solution: this is one place where the point of view of affine schemes as a subcategory of locally ringed spaces is useful, as one can show that one can "glue together" locally ringed spaces. So one can define schemes as the subcategory of locally ringed spaces whose objects are glued together affine schemes.
- Functor of points solution: since we have affine schemes = cAlg_R^{op}, we can also think of affine schemes as the full subcategory of the associated functor category $[\text{cAlg}_R, set]$ just consisting of the representable functors (which is equivalent to cAlg_R^{op} by Yoneda). But the larger functor category $[\text{cAlg}_R, set]$ has all colimits, and one can make precise exactly which functors in $[\text{cAlg}_R, set]$ are those which are "glued together affine schemes" - eg., see Eisenbud/Harris Theorem VI-14.
- Restriction categories solution: one can build a restriction category from cAlg_R^{op} with the restriction monics consisting of the monomorphisms $A_f \hookrightarrow A$ (for $f \in A$). Then the category of schemes should be the manifold completion of the join completion of this restriction category.

5 Comparisons and look forward

5.1 Comparison to the standard "locally ringed space" approach

Now that we've seen one way to approach modern algebraic geometry, it would be useful to compare what has been described here to the standard done in most textbooks. In the standard approach, there are several steps to building the "affine scheme" associated to a commutative unital ring A:

- One begins by building a "set of points" of A: these are defined to be the prime ideals (!!) of A; write this set as $\mathrm{Spec}(A)$.
- Then one defines a topology on $\mathrm{Spec}(A)$, which has as basis the sets

  $$D(f) := \{P \in \mathrm{Spec}(A) : f \notin P\}$$

  for each $f \in A$.
- Next, one defines a sheaf $\mathcal{O}_A$ on this topological space which can be defined by its action on the basis elements:

  $$\mathcal{O}_A(D(f)) := A_f$$

  (the localization of the ring A at the multiplicative set generated by f).
- The stalk of this sheaf at a point $P \in \mathrm{Spec}(A)$ is A_P (the localization of the ring A at the complement of the prime ideal P). This is a local ring (= has a unique maximal ideal, in this case P itself becomes a maximal ideal in A_P).

In its entirety, this associates to each ring R a "locally ringed space" $(\mathrm{Spec}(A), \mathcal{O}_A)$, where a locally ringed space in general consists of a topological space and a sheaf of rings on that topological space such that the stalk of the sheaf at each point is a local ring.

Morphisms of these objects are morphisms of locally ringed spaces, which are continuous functions which preserve the sheaf structure and the stalks appropriately. By definition, the category of affine schemes is the full subcategory of locally ringed spaces consisting of locally ringed spaces of the form $\mathrm{Spec}(A)$.

Key result on affine schemes: the category of affine schemes is equivalent to the opposite of the category of commutative rings (and then one can take slices to get schemes over a fixed base, eg., affine schemes over R = commutative rings opposite over R = (R over commutative rings) opposite = (commutative R-algebras) opposite).

One way to see what is being done here is that the locally ringed space point of view is taking certain aspects of the category $\mathbf{cAlg}_R^{op}$ and making the maps into objects as explicit data of these objects themselves. For example, in earlier sections, we described how maps from various objects corresponded to the right notion of "point" of an algebraic variety A eg., maps from the terminal object R were R-points, maps from $S \supset R$ were S-points, and generic points were maps from the field of fractions of quotients A. In the above point of view, many of these "points" (the ones whose domain are fields) are lumped together into a single set by defining a "point" as simply a prime ideal. One should think of the point $P \in \mathrm{Spec}(A)$ as corresponding to the morphism

$$k(P) := (A/P)^{ff} \longrightarrow A$$

(where ff is the field of fractions = localize all non-zero elements). As noted above, this includes all sorts of types of "points". For example, if we take $R = \mathbb{R}$ and $A = R[x, y]$, then this includes:

- The "ordinary" $\mathbb{R}$ points, eg., $P = (x-3)(y-2)$, corresponding to the geometric map
$$(\mathbb{R}[x, y]/(x-3)(y-2))^{ff} \cong \mathbb{R} \longrightarrow \mathbb{R}[x, y]$$
corresponding to the point $(3, 2)$ of affine 2-space.
- The "imaginary" $\mathbb{C}$ points, eg., if $P = (x^2+1)(y)$, then we get
$$(\mathbb{R}[x, y]/(x^2+1)(y))^{ff} \cong \mathbb{C} \longrightarrow R[x, y]$$
Something interesting happens here: depending on whether we identify $\mathbb{R}[x, y]/(x^2+1)(y)$ with $\mathbb{C}$ with $x \mapsto i$ or $x \mapsto -i$, we get a different complex point (either $(i, 0)$ or $(-i, 0)$). But there is just one prime ideal. So in this version, a "point" actually identifies complex points satisfying the same equation.
- The "generic" points, eg., if $P = (y^2 - x^3)$, then we get
$$(R[x, y]/(y^2 - x^3))^{ff} \longrightarrow R[x, y]$$
which we described earlier as the "generic point of the curve $y^2 = x^3$".

So, in the locally ringed space point of view, all these (quite different!) types of "points" are lumped together. This can lead to some weird artifacts: if you take the product of two (affine) schemes X and Y, in general the points of $X \times Y$ need not be the set of points of X times the points of Y. In a similar vein, Deligne himself[3] writes

> The points of Spec(A)...have no ready to hand geometric sense...When one needs to construct a scheme one generally does not begin by constructing the set of points.

The topology one builds in $\mathrm{Spec}(A)$ has similar issues. The topology itself is generally not interesting; as almost all textbooks note, it's open sets are "too large". This then gives signficant issues when trying to use cohomology: the cohomology relative to the Zariski topology of a scheme in general contains almost no non-trivial information. (Hence why Grothendieck and others need to build other "topologies" on which to build a cohomology theory for schemes on).

So, is there value in the classical approach? Certainly it's the view almost universally done in algebraic geometry textbooks, but of course just because things have always been done this way doesn't mean it's the best way to do things! But it certainly can be useful in various ways:

[3] In Mclarty's translation in his article "The Rising Sea: Grothendieck on simplicity and generality I"

- Perhaps the most important is the nearly all algebraic geometry articles are written from this point of view! So any introduction to the subject must at least talk about this point of view.
- The development of the locally ringed space associated to a commutative ring mimics how people arrived at the idea historically, and this can sometimes be helpful to trace the development piece by piece...for example, the idea grew from the Nullstensatz that certain points corresponded to maximal ideals...thinking about other fields and rings led to the generalization to prime ideals...the realization of a "topology" on this set brought the idea in closer contact to the newly developing field of topology...the development of a natural sheaf on this topological space was related to sheaf theories which were developing around that time.
- As noted in Section 4.5, the category of locally ringed spaces "has all gluings", so it's a natural place to move from affine schemes to schemes (= gluings of affine schemes).
- There is surely some significant value in the statement of the "key result on affine schemes", even if one does start with the point of view described in this rest of these notes. In general, in any category, the Yoneda Lemma tells you that each object A is determined by all maps into A. But in general this is a huge collection of information, so if one can find a subcollection of these maps which determine the object, this is always useful. And the locally ringed space point of view does this, essentially showing that an affine scheme is determined by its "points" (maps from fields), it's "open subsets" (localizations at a point), and its "stalks" (localization at the complement of a prime ideal).

All this being said, to my mind, the optimal way to present this material (especially if one's audience is familiar with category theory) is to *first* introduce the geometric category and see how it really is geometric (eg., maps from terminal object = elements of the variety, pullbacks = intersections, fibres, etc.) then only later describe the locally ringed space approach, as has been done here.